

Generalized Bockstein maps and Massey products

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Abstract

Given a profinite group G of finite p -cohomological dimension and a pro- p quotient H of G by a closed normal subgroup N , we study the filtration on the Iwasawa cohomology of N by powers of the augmentation ideal in the group algebra of H . We show that the graded pieces are related to the cohomology of G via analogues of Bockstein maps for the powers of the augmentation ideal. For certain groups H , we relate the values of these generalized Bockstein maps to Massey products relative to a restricted class of defining systems depending on H . We apply our study to give a new proof of the vanishing of triple Massey products in Galois cohomology.

1 Introduction

Let G be a profinite group of p -cohomological dimension 2, and let H be a finitely generated pro- p quotient of G by a closed normal subgroup N . Let $\Omega = \mathbb{Z}_p[[H]]$ denote the completed \mathbb{Z}_p -group ring of H , the inverse limit of the \mathbb{Z}_p -group rings of the finite quotients of H . Let T be a finitely generated \mathbb{Z}_p -module with a continuous action of G . This paper is concerned with the study of connecting maps in the G -cohomology of the augmentation filtration of the tensor product $T \otimes_{\mathbb{Z}_p} \Omega$. That is, if $I = \ker(\Omega \rightarrow \mathbb{Z}_p)$ denotes the augmentation ideal of Ω , then we have exact sequences

$$0 \rightarrow T \otimes_{\mathbb{Z}_p} I^n / I^{n+1} \rightarrow T \otimes_{\mathbb{Z}_p} \Omega / I^{n+1} \rightarrow T \otimes_{\mathbb{Z}_p} \Omega / I^n \rightarrow 0 \quad (1.1)$$

for each $n \geq 1$ such that Ω / I^n is \mathbb{Z}_p -flat. We study the connecting homomorphisms

$$\Psi^{(n)}: H^1(G, T \otimes_{\mathbb{Z}_p} \Omega / I^n) \rightarrow H^2(G, T) \otimes_{\mathbb{Z}_p} I^n / I^{n+1}$$

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attached to these sequences, which we refer to as *generalized Bockstein maps*, due to their similarity to usual Bockstein maps for exact sequences of p -power order cyclic groups.

We can use the Bockstein maps to partially describe the second *Iwasawa cohomology group* $H_{\text{Iw}}^2(N, T)$ of N with T -coefficients. This cohomology group is the inverse limit of the groups $H^2(U, T)$ under corestriction maps, where U runs over the open normal subgroups of G containing N . It is naturally endowed, through the $\mathbb{Z}_p[G/U]$ -actions on each $H^2(U, T)$, with the structure of an Ω -module. We prove that the cokernels of the generalized Bockstein maps describe the graded quotients in the augmentation filtration of $H_{\text{Iw}}^2(N, T)$ (see Theorem 2.2.4).

Theorem A. *There are canonical isomorphisms*

$$\frac{I^n H_{\text{Iw}}^2(N, T)}{I^{n+1} H_{\text{Iw}}^2(N, T)} \cong \frac{H^2(G, T) \otimes_{\mathbb{Z}_p} I^n / I^{n+1}}{\text{im } \Psi^{(n)}}.$$

The proof rests on an Iwasawa-cohomological version [LiSh, FuKa] of a descent spectral sequence of Tate, applied to the terms of our exact sequences for the augmentation filtration of Ω . We verify the compatibility of these spectral sequences with generalized Bockstein maps and a connecting map in the H -homology of the \mathbb{Z}_p -tensor product of $H_{\text{Iw}}^2(N, T)$ with (1.1).

The case $H \cong \mathbb{Z}_p$ was first studied in [Sh1] from a different perspective and applied in an Iwasawa-theoretic context. Its main result has a similar form to Theorem A, but in place of the image of $\Psi^{(n)}$, it has a group of values of certain $(n + 1)$ -fold *Massey products*, which are higher operations in Galois cohomology generalizing cup products. We describe threefold Massey products after the statement of Theorem B below.

Here, we show that in the case that $H \cong \mathbb{Z}_p$, the values of generalized Bockstein maps are in fact values of certain Massey products relative to the *proper defining systems* introduced in [Sh1]. These systems reduce the indeterminacy inherent in the definition of a Massey product, which is only defined if certain lower Massey products vanish, and if defined still depends on the choice of a *defining system*, a homomorphism from G to a group of unipotent matrices modulo upper right-hand corners.

This raises the question of whether one can relate the values of generalized Bockstein maps and Massey products for more general groups H , especially, beyond the procyclic setting of [Sh1]. We shall discuss three classes of groups for which the answer is yes, the torsion-free groups in these being \mathbb{Z}_p , \mathbb{Z}_p^2 and the Heisenberg group over \mathbb{Z}_p .

The most difficult step in making the comparison is to obtain notions of proper defining systems of Massey products with values relating to projections of the values of the generalized Bockstein maps corresponding to elements of a \mathbb{Z}_p -basis of I^n / I^{n+1} , which is \mathbb{Z}_p -free. For this, we start with a *partial defining system*, which consists of a pair of homomorphisms from

H to upper-triangular unipotent matrices of dimensions $a + 1$ and $b + 1$, where $a + b = n$. Our proper defining systems are extensions of these pairs viewed as maps on G to a defining system, with the two maps giving the two diagonal blocks in the defining system: see Definition 3.3.2.

For example, we prove the following result for $H \cong \mathbb{Z}_p^2$ in Theorem 4.4.3.

Theorem B. *Suppose that $\nu = (\chi, \psi): H \rightarrow \mathbb{Z}_p^2$ is an isomorphism. Let $x, y \in I$ be such that $x + 1$ and $y + 1$ are group elements mapping under ν to the standard ordered basis of \mathbb{Z}_p^2 . For $n \geq 2$, the cosets of $x^a y^{n-a}$ with $0 \leq a \leq n$ then form a \mathbb{Z}_p -basis for I^n / I^{n+1} .*

a. *To a 1-cocycle $f: G \rightarrow T \otimes_{\mathbb{Z}_p} \Omega / I^n$ and $0 \leq a \leq n$, we can associate a proper defining system for an $(n + 1)$ -fold Massey product*

$$(\chi^{(a)}, \lambda, \psi^{(n-a)}) := (\underbrace{\chi, \dots, \chi}_{a \text{ times}}, \lambda, \underbrace{\psi, \dots, \psi}_{n-a \text{ times}}) \in H^2(G, T),$$

where $\lambda: G \rightarrow T$ is the composition of f with the quotient map $T \otimes_{\mathbb{Z}_p} \Omega / I^n \rightarrow T \otimes_{\mathbb{Z}_p} \Omega / I \cong T$.

b. *With the notation of part a, let $[f]$ denote the class of f in $H^1(G, T \otimes_{\mathbb{Z}_p} \Omega / I^n)$. Then*

$$\Psi^{(n)}([f]) = \sum_{a=0}^n (\chi^{(a)}, \lambda, \psi^{(n-a)}) \otimes x^a y^{n-a}.$$

Let us illustrate Theorem B in some detail in the case that $n = 2$ and $a = 1$. In this case,

$$\Omega / I^2 = \mathbb{Z}_p[x, y] / (x^2, xy, y^2)$$

in the notation of the theorem. We can therefore write the 1-cocycle $f: G \rightarrow T \otimes_{\mathbb{Z}_p} \Omega / I^2$ as

$$f = \lambda + \lambda_x x + \lambda_y y,$$

with $\lambda_x, \lambda_y: G \rightarrow T$, abbreviating the tensor product as formal multiplication. Part a of Theorem B says that f gives rise to a defining system

$$\rho = \left(\begin{array}{cc|cc} 1 & \chi & \lambda_x & * \\ & 1 & \lambda & \lambda_y \\ & & 1 & \psi \\ & & & 1 \end{array} \right) : G \rightarrow U/Z$$

for the Massey triple product (χ, λ, ψ) . Here, the values of ρ lie in the quotient of a group U of *generalized* upper-triangular unipotent 4-by-4 matrices by its subgroup Z of matrices

with zero above-diagonal entries outside of the upper right-hand corner. The entries in the upper-right hand block are T -valued (and in particular $Z \cong T$), whereas they are \mathbb{Z}_p -valued outside of it. Matrix multiplication proceeds using the \mathbb{Z}_p -module structure on T . That ρ is a defining system means that $\rho: G \rightarrow U/Z$ is a nonabelian 1-cocycle, where G acts on U coordinate-wise. The Massey product $(\chi, \lambda, \psi)_\rho$ relative to the defining system ρ is an element of $H^2(G, T)$ providing the obstruction to lifting ρ to a nonabelian 1-cocycle $G \rightarrow U$.

In general, even for such a cocycle ρ and therefore a Massey product (χ, λ, ψ) to exist, the cup products $\chi \cup \lambda$ and $\lambda \cup \psi$ must vanish in $H^2(G, T)$ so that cochains λ_x and λ_y can be chosen with $d\lambda_x = -\chi \cup \lambda$ and $d\lambda_y = -\lambda \cup \psi$. Even then, the class (χ, λ, ψ) depends on these choices. In our description, this vanishing is encapsulated in the fact that f is a 1-cocycle, and the indeterminacy is removed by fixing f .

The content of part b of Theorem B is that the coefficients of $\Psi^{(2)}([f])$ in $H^2(G, T)$ for the \mathbb{Z}_p -basis x^2 , xy , and y^2 of I^2/I^3 are triple Massey products: in particular, the coefficient of xy is the Massey product $(\chi, \lambda, \psi)_\rho$ for the defining system ρ . More precisely, $(\chi, \lambda, \psi)_\rho$ is defined as class of the 2-cocycle $F: G^2 \rightarrow T$ given by

$$F: (g, h) \mapsto \chi(g)g\lambda_x(h) + \psi(h)\lambda_y(g).$$

This cocycle F arises as the upper-right hand corner of $(g, h) \mapsto \tilde{\rho}(g) \cdot g\tilde{\rho}(h)$ for the naive lift of ρ to a cochain $\tilde{\rho}: G \rightarrow U$ with zero upper-right hand corner. The theorem boils down to the fact that $F \cdot xy$ is also exactly the coboundary of the naive lift of f to a cochain $G \rightarrow \mathbb{Z}_p[x, y]/(x^2, y^2)$ with zero xy -coefficient.

From our perspective, the generalized Bockstein maps are more flexible than Massey products, being connecting homomorphisms more directly amenable to basic applications of homological algebra. For instance, the argument proving Theorem A for arbitrary H amounts to a diagram chase for maps of Grothendieck spectral sequences. Moreover, Theorem B allows us to study defining systems using abelian, rather than nonabelian, cocycles.

At its core, our work is motivated by the potential arithmetic and Galois-theoretic applications. In the preprint [Sh3], the results of this paper are applied in the setting of Iwasawa theory to study inverse limits of class groups. In that case, G is the Galois group of the maximal extension of a number field unramified outside a finite set of primes containing those above p , and H is the Galois group of a \mathbb{Z}_p -extension. The group $H_{\text{Iw}}^2(N, \mathbb{Z}_p(1))$ is closely related to but not always isomorphic to the usual inverse limit X of class groups up the tower defined by H . The isomorphisms of Theorem A are then used to derive exact sequences for the graded pieces in the augmentation filtration of X .

In the final section of this paper, we apply our techniques to study absolute Galois groups of fields. The motivating problem is to determine which profinite groups can be isomorphic

to the absolute Galois group G_F of a field F . Artin and Schreier showed in 1927 that any nontrivial finite group with this property is the cyclic group has order two. Other restrictions are reflected in the cohomological properties of G_F . The norm residue isomorphism theorem, or Milnor-Bloch-Kato conjecture, proven by Voevodsky and Rost (see [Vo]), tells us that the algebra $H^*(G_F, \mathbb{F}_p)$ under cup product is isomorphic to the mod- p Milnor K -theory of F (for F containing a primitive p th root of 1). In particular, this implies that the \mathbb{F}_p -cohomology algebra is generated in degree 1 with all relations generated in degree 2.

Going beyond cup products to higher cohomological operations, Mináč and Tân formulated a remarkable conjecture, known as the *Massey vanishing conjecture*, for Massey products of \mathbb{F}_p -valued characters on the absolute Galois group G_F of a field F in [MT4]. For $n \geq 3$, it states that any n -fold Massey product of characters $G_F \rightarrow \mathbb{F}_p$ that has a defining system has some defining system for which the resulting Massey product is zero. The Massey product is said to *contain zero* if such a defining system exists.

Prior to the formulation of this conjecture, Hopkins and Wickelgren had already proved using splitting varieties in [HoWi] that for number fields, n -fold Massey products that are defined contain zero for $n = 3$ and $p = 2$. Their work was later generalized to arbitrary p and successively general n : to $n = 3$ in [MT2], to $n = 4$ in [GMT], and in a fairly recent preprint [HrWt] to arbitrary n , all over number fields.

For arbitrary fields F , Efrat–Matzri [EfMa] and Mináč–Tân [MT3] independently proved *triple Massey vanishing*, which is to say the conjecture for $n = 3$ and arbitrary p . Other results tend to require that several of the characters in the Massey products be the same. For instance, the third author had long ago proved in [Sh1] what we refer to here as the *p -cyclic Massey vanishing property* for absolute Galois groups: for $n \leq p - 1$, all $(n + 1)$ -fold Massey products $(\chi^{(n)}, \psi)$ for $\chi, \psi \in H^1(G_F, \mathbb{F}_p)$ with $\chi \cup \psi = 0$ vanish with respect to some proper defining system. More recently, Mináč and Tân [MT1] proved the vanishing of n -fold Massey products when all n characters are the same, for arbitrary n .

To illustrate the use of our constructions, we apply our theory of generalized Bockstein maps to give new, streamlined proofs of both the p -cyclic Massey vanishing property and triple Massey vanishing. In fact, we show the following in Theorem 5.2.1.

Theorem C. *Let G be a profinite group with the p -cyclic Massey vanishing property for an odd prime p . Then every triple Massey product on $H^1(G, \mathbb{F}_p)$ which is defined contains zero.*

In our proof of Theorem C, to show that a defined Massey product (χ, λ, ψ) vanishes, we consider the coimage H of the map $(\chi, \psi): G \rightarrow \mathbb{F}_p^2$. We then apply a variant of Theorem B to this H to see that the Massey product (χ, λ, ψ) relative to a certain defining system is the obstruction to lifting λ to a class in $H^1(G, \Omega/J)$ for a particular ideal J between I^2 and

I^3 . Via an involved diagram chase, we see that the p -cyclic Massey vanishing property for the quotients of H that are the coimages of χ , ψ , and $\chi + \psi$ implies that this obstruction equals $\nu \cup (\chi + \psi)$ for some $\nu \in H^1(G, \mathbb{F}_p)$. This is enough to show that the Massey product contains zero.

Theorem C raises several interesting questions that we do not attempt to address here, including whether or not the vanishing of Massey products $(\chi^{(n)}, \psi)$ for arbitrary n is sufficient to imply Massey vanishing.

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2 Generalized Bockstein maps

In this section, we define generalized Bockstein maps and employ them in the study of the structure of inverse limits of cohomology groups. Throughout, we use the following objects:

- a profinite group G ,
- a topologically finitely generated pro- p quotient H of G by a closed normal subgroup N ,
- a compact Noetherian \mathbb{Z}_p -algebra R (usually taken to be a quotient of \mathbb{Z}_p),
- the completed group ring $\Omega = R[[H]]$,
- the augmentation ideal I of Ω , i.e., the kernel of the continuous R -algebra homomorphism $\Omega \rightarrow R$ that sends every group element in H to 1,
- a positive integer n such that Ω/I^n and I^n/I^{n+1} are R -flat, and
- a compact $R[[G]]$ -module T that is R -finitely generated.

Note that a compact $R[[G]]$ -module is the same as a compact R -module with a continuous R -linear action of G . We will frequently take tensor products $M \otimes_R M'$ of compact $R[[G]]$ -modules M and M' , at least one of which is finitely generated over R . These compact R -modules (with the topology of the isomorphic completed tensor product) have the diagonal action of G .

2.1 Augmentation sequences

Since we have assumed that Ω/I^n is R -flat, the right exact sequence of compact $R[[G]]$ -modules

$$0 \rightarrow T \otimes_R I^n/I^{n+1} \rightarrow T \otimes_R \Omega/I^{n+1} \rightarrow T \otimes_R \Omega/I^n \rightarrow 0 \quad (2.1)$$

is exact. For any $d \geq 1$, we have the resulting connecting homomorphisms

$$H^{d-1}(G, T \otimes_R \Omega/I^n) \rightarrow H^d(G, T \otimes_R I^n/I^{n+1})$$

on continuous G -cohomology.

Since G acts trivially on the finitely generated R -module I^n/I^{n+1} , we have a homomorphism

$$H^d(G, T) \otimes_R I^n/I^{n+1} \rightarrow H^d(G, T \otimes_R I^n/I^{n+1}) \quad (2.2)$$

that is an isomorphism as I^n/I^{n+1} is R -flat, so long as we assume either that G has finite p -cohomological dimension or that I^n/I^{n+1} has a finite resolution by projective R -modules (see [LiSh, Proposition 3.1.3], the proof of which does not use the assumption on R in that section). The latter condition is automatic, given that I^n/I^{n+1} is flat, if R is a quotient of \mathbb{Z}_p . We let

$$\Psi^{(n)}: H^{d-1}(G, T \otimes_R \Omega/I^n) \rightarrow H^d(G, T) \otimes_R I^n/I^{n+1}. \quad (2.3)$$

denote the resulting composite map, and we refer to it as a *generalized Bockstein map*.

Remark 2.1.1. We may replace the assumption that Ω/I^n is R -flat with the assumption that T is R -flat in order that (2.1) still hold. We may also replace the assumption that I^n/I^{n+1} is R -flat with the assumption that G has p -cohomological dimension d and still have an isomorphism as in (2.2). (To see this, choose a presentation of I^n/I^{n+1} by finitely generated free R -modules and use the right exactness of the d th cohomology functor and the tensor product, noting that $H^d(G, T^r) \cong H^d(G, T) \otimes_R R^r$ for any r .) With either replacement, $\Psi^{(n)}$ is still a map as in (2.3).

2.2 Graded quotients of Iwasawa cohomology groups

Recall that N denotes the kernel of the surjection $G \rightarrow H$. Our interest in this section is in the *Iwasawa cohomology groups*

$$H_{\text{Iw}}^i(N, T) = \varprojlim_{N \leq U \trianglelefteq G} H^i(U, T)$$

for $i \geq 1$, where the inverse limit is taken with respect to corestriction maps over open normal subgroups U of G containing N . Note that the Iwasawa cohomology groups are relative to the larger group G , though this is omitted from our notation. Since each $H^i(U, T)$ is a $R[G/U]$ -module and the actions are compatible with corestriction, the group $H_{\text{Iw}}^i(N, T)$ is endowed with the structure of an Ω -module.

Remark 2.2.1. If H is finite, then $H_{\text{Iw}}^i(N, T) = H^i(N, T)$.

Let us define two notions that we need. First, a profinite group \mathcal{G} is *p -cohomologically finite* if \mathcal{G} has finite p -cohomological dimension and $H^i(\mathcal{G}, M)$ is finite for every finite $\mathbb{Z}_p[\mathcal{G}]$ -module M and $i \geq 0$. Second, a *compact p -adic Lie group* is a profinite group that has an open pro- p subgroup any closed subgroup of which can be topologically generated by r elements for some fixed r . Equivalently, a compact p -adic Lie group is any profinite group continuously isomorphic to a closed subgroup of $\text{GL}_n(\mathbb{Z}_p)$ for some $n \geq 1$.

We make the following assumptions for the rest of this section:

- G is p -cohomologically finite of p -cohomological dimension d ,
- R is a complete commutative local Noetherian \mathbb{Z}_p -algebra with finite residue field, and
- either
 - (i) H is a compact p -adic Lie group or
 - (ii) T has a finite resolution by a complex of $R[[G]]$ -modules free of finite rank over R .

Recall that the zeroth H -homology group of a compact Ω -module M is its coinvariant module $M_H \cong M/IM$. In our setting, corestriction provides an isomorphism on coinvariants in degree d (see [NSW, Proposition 3.3.11]), which is to say that we have a natural isomorphism

$$\frac{H_{\text{Iw}}^d(N, T)}{IH_{\text{Iw}}^d(N, T)} \cong H^d(G, T). \quad (2.4)$$

This gives rise to a Grothendieck spectral sequence for the implicit composition of right exact functors, which is a version of Tate's descent spectral sequence for Iwasawa cohomology

Proposition 2.2.2 (Fukaya-Kato, Lim-Sharifi). *The Ω -modules $H_{\text{Iw}}^i(N, T)$ are finitely generated for all $i \geq 0$. Moreover, we have a first quadrant homological spectral sequence of R -modules*

$$E_{i,j}^2(T) = H_i(H, H_{\text{Iw}}^{d-j}(N, T)) \Rightarrow E_{i+j}(T) = H^{d-i-j}(G, T),$$

where d is the p -cohomological dimension of G .

This result is proven in [Ta, Theorem 1] if H is finite, and it follows from [FuKa, Proposition 1.6.5] if (ii) holds and from [LiSh, Propositions 3.1.3 and 3.2.4] if (i) holds.

The isomorphism (2.4) and the other edge maps on coinvariant groups in this spectral sequence are given by the inverse limits of corestriction maps. This isomorphism forces the n th graded quotient $I^n A / I^{n+1} A$ in the augmentation filtration of $A = H_{\text{Iw}}^d(N, T)$ to be a quotient of $H^d(G, T) \otimes_R I^n / I^{n+1}$ using the surjective map

$$A / IA \otimes_R I^n / I^{n+1} \rightarrow I^n A / I^{n+1} A$$

induced by the map $A \times I^n \rightarrow I^n A$ given by the multiplication $(a, x) \mapsto xa$. As we shall see, this quotient is in fact $\text{coker } \Psi^{(n)}$.

Recall that we have assumed that Ω / I^n is R -flat. Moreover, the fact that H is topologically finitely generated implies that Ω / I^n is finitely generated over R .

Lemma 2.2.3. *Let A be an Ω -module, and consider the exact sequence*

$$0 \rightarrow A \otimes_R I^n / I^{n+1} \rightarrow A \otimes_R \Omega / I^{n+1} \rightarrow A \otimes_R \Omega / I^n \rightarrow 0. \quad (2.5)$$

The connecting homomorphism

$$\partial_n : H_1(H, A \otimes_R \Omega / I^n) \rightarrow A_H \otimes_R I^n / I^{n+1}$$

in the H -homology of (2.5) has cokernel isomorphic to $I^n A / I^{n+1} A$.

Proof. We have compatible, natural isomorphisms of R -modules

$$(A \otimes_R \Omega / I^m)_H \cong \Omega / I^m \otimes_{\Omega} A \cong A / I^m A$$

for $m \geq 1$ given on $a \in A$ and $\omega \in \Omega$ (or its quotient by I^m) by

$$a \otimes \omega \mapsto \iota(\omega) \otimes a \mapsto \iota(\omega)a,$$

where $\iota : \Omega \rightarrow \Omega$ is the unique continuous R -linear map given by inversion of group elements on H . Note that the switch of terms in the tensor product in the first isomorphism is necessitated

by the fact that A is a left Ω -module. (In fact, these become isomorphisms of Ω -modules since $a \otimes \omega h^{-1} \mapsto h \cdot \iota(\omega) \otimes a$ for $h \in H$ under the first map.)

By the long exact sequence in H -homology and the above isomorphisms, the cokernel of interest is identified with the kernel of the quotient map $A/I^{n+1}A \rightarrow A/I^n A$, hence the result. \square

We now come to our theorem.

Theorem 2.2.4. *For each $n \geq 1$, there is a canonical isomorphism*

$$\frac{I^n H_{\text{Iw}}^d(N, T)}{I^{n+1} H_{\text{Iw}}^d(N, T)} \cong \frac{H^d(G, T) \otimes_R I^n / I^{n+1}}{\text{im } \Psi^{(n)}}$$

of R -modules, where d is the p -cohomological dimension of G .

Proof. There are isomorphisms

$$H_{\text{Iw}}^d(N, T \otimes_R M) \cong H_{\text{Iw}}^d(N, T) \otimes_R M$$

for any compact $R[[G]]$ -module M finitely generated over R , since G has p -cohomological dimension d . In particular, the following sequence is exact:

$$0 \rightarrow H_{\text{Iw}}^d(N, T \otimes_R I^n / I^{n+1}) \rightarrow H_{\text{Iw}}^d(N, T \otimes_R \Omega / I^{n+1}) \rightarrow H_{\text{Iw}}^d(N, T \otimes_R \Omega / I^n) \rightarrow 0.$$

We consider the connecting homomorphism in H -homology:

$$\partial^{(n)}: H_1(H, H_{\text{Iw}}^d(N, T) \otimes_R \Omega / I^n) \rightarrow H_{\text{Iw}}^d(N, T)_H \otimes_R I^n / I^{n+1}. \quad (2.6)$$

We next apply Lemma A.0.1 of the appendix, which says that edge maps to total terms in homological Grothendieck spectral sequences are compatible with connecting maps. Here, the spectral sequence is that of Proposition 2.2.2, which is associated to the composition of functors $F = H_0(H, \cdot)$ and $F' = H_{\text{Iw}}^d(N, \cdot)$, noting that $F \circ F' \cong H^d(G, \cdot)$ via corestriction. The connecting homomorphisms are from degrees 1 to 0 and are associated to the short exact sequence of (2.1).

In this setting, the lemma provides a commutative square related to the diagram

$$\begin{array}{ccc} H^{d-1}(G, T \otimes_R \Omega / I^n) & \xrightarrow{\Psi^{(n)}} & H^d(G, T) \otimes_R I^n / I^{n+1} \\ \downarrow & & \downarrow \wr \\ H_1(H, H_{\text{Iw}}^d(N, T) \otimes_R \Omega / I^n) & \xrightarrow{\partial^{(n)}} & H_{\text{Iw}}^d(N, T)_H \otimes_R I^n / I^{n+1}, \end{array} \quad (2.7)$$

but with $L_1(F \circ F')(T \otimes_R \Omega / I^n)$ in place of $H^{d-1}(G, T \otimes_R \Omega / I^n)$. By Lemma A.0.2, which is a simple consequence of the universality of left derived functors, we have a surjection from the

latter object to the former compatible with their connecting homomorphisms to $H^d(G, T) \otimes_R I^n/I^{n+1}$. This allows us to make the replacement while maintaining the surjectivity of the left vertical map, so we indeed have the commutative square (2.7).

By Lemma 2.2.3, the isomorphism in the statement of the theorem is the map on cokernels of the horizontal maps in (2.7). \square

Although not used in this paper, for the purposes of Iwasawa-theoretic applications, it is useful to have a slightly stronger version of Theorem 2.2.4. So, we remark that it has the following generalization, with virtually no additional complications (given that the results of [LiSh, FuKa] hold in this generality).

Remark 2.2.5. Let \mathcal{G} be a profinite group, and let Γ be a quotient of \mathcal{G} by a closed normal subgroup G . Let \mathcal{H} be a quotient of \mathcal{G} by a closed normal subgroup N that is contained in G , and let $H = G/N$ as before. We then have $\Gamma \cong \mathcal{H}/H$. That is, we have a commutative diagram of exact sequences

$$\begin{array}{ccccc} N & \xlongequal{\quad} & N & & \\ \downarrow & & \downarrow & & \\ G & \hookrightarrow & \mathcal{G} & \twoheadrightarrow & \Gamma \\ \downarrow & & \downarrow & & \parallel \\ H & \hookrightarrow & \mathcal{H} & \twoheadrightarrow & \Gamma. \end{array}$$

Take T to be a compact $R[[\mathcal{G}]]$ -module finitely generated over R , and replace the assumptions on G and H from the beginning of this subsection with the identical assumptions on \mathcal{G} and \mathcal{H} , respectively. We have Iwasawa cohomology groups $H_{\text{Iw}}^i(N, T)$ and $H_{\text{Iw}}^i(G, T)$, which are now taken relative to the larger group \mathcal{G} . These are finitely generated as modules over $R[[\mathcal{H}]]$ and $\Lambda = R[[\Gamma]]$, respectively, and we have as before a spectral sequence

$$E_{i,j}^2(T) = H_i(H, H_{\text{Iw}}^{d-j}(N, T)) \Rightarrow E_{i+j}(T) = H_{\text{Iw}}^{d-i-j}(G, T),$$

but now of Λ -modules. In exactly the same manner as before, this gives rise to isomorphisms

$$\frac{I^n H_{\text{Iw}}^d(N, T)}{I^{n+1} H_{\text{Iw}}^d(N, T)} \cong \frac{H_{\text{Iw}}^d(G, T) \otimes_R I^n/I^{n+1}}{\text{im } \Psi^{(n)}},$$

again of Λ -modules.

2.3 The abelian case

We turn to the direct computation of generalized Bockstein maps on 1-cocycles for abelian H . That is, let us now take H to be a finitely generated, abelian pro- p group, and let us take R to

be a quotient of \mathbb{Z}_p . We give an explicit formula for $\Psi^{(n)}$ under a hypothesis on the size of R that ensures our flatness hypothesis is satisfied. If H has no nonzero p -torsion, no hypothesis is needed.

We begin with the following simple lemma.

Lemma 2.3.1. *Let s and t be positive integers with $n < p^{t-s+1}$. Then $(1+x)^{p^t} - 1$ is in the ideal (x^{n+1}, p^s) of $\mathbb{Z}[x]$.*

Proof. Recall that p^s divides $\binom{p^t}{i}$ for $0 < i < p^{t-s+1}$. Therefore

$$(1+x)^{p^t} = \sum_{i=0}^{p^t} \binom{p^t}{i} x^i \equiv 1 \pmod{(x^{n+1}, p^s)}$$

so long as $n < p^{t-s+1}$. □

Let h_1, \dots, h_r be a minimal set of generators for H , labeled such that h_1, \dots, h_c have finite orders $p^{t_1} \leq \dots \leq p^{t_c}$ and h_{c+1}, \dots, h_r have infinite order, for some $0 \leq c \leq r$. Define $x_i = [h_i] - 1 \in \Omega$ for $1 \leq i \leq r$, where $[h_i]$ denotes the group element of h_i , so that $I = (x_1, \dots, x_r)$. We then have

$$\Omega \cong \frac{R[[x_1, \dots, x_r]]}{((x_1 + 1)^{p^{t_1}} - 1, \dots, (x_c + 1)^{p^{t_c}} - 1)}.$$

We have $c > 0$ if and only if H is not \mathbb{Z}_p -free, in which case we suppose that $R = \mathbb{Z}/p^s\mathbb{Z}$ with $n < p^{t_1-s+1}$. By Lemma 2.3.1, we have

$$\Omega/I^j \cong \frac{R[x_1, \dots, x_r]}{(x_1, \dots, x_r)^j}$$

for $j \leq n+1$. Moreover, I^n/I^{n+1} is a free R -module with basis the monomials in the variables x_i of degree n . In particular, the generalized Bockstein map $\Psi^{(n)}$ is defined.

We may view any element of $q \in T \otimes_R \Omega/I^n$ as having the form

$$q = \sum_{k_1 + \dots + k_r < n} \alpha_{k_1, \dots, k_r} x_1^{k_1} \cdots x_r^{k_r},$$

where the sum is taken over r -tuples (k_1, \dots, k_r) of nonnegative integers with sum less than n and with $\alpha_{k_1, \dots, k_r} \in T$, omitting the notation for the tensor product in such an expression. Setting $\|k\| = k_1 + \dots + k_r$ for an r -tuple (k_1, \dots, k_r) , let's simplify this notation as

$$q = \sum_{\|k\| < n} \alpha_k x^k, \tag{2.8}$$

where $x^k = x_1^{k_1} \cdots x_r^{k_r}$.

Let $\pi: G \rightarrow H$ denote the quotient map. For each i , let

$$A_i = \begin{cases} \mathbb{Z}/p^{t_i}\mathbb{Z} & \text{if } 1 \leq i \leq c, \\ \mathbb{Z}_p & \text{if } c < i \leq r. \end{cases}$$

For $1 \leq i \leq r$, let $\chi_i: G \rightarrow A_i$ be the homomorphisms determined by

$$\pi(g) = \prod_{i=1}^r h_i^{\chi_i(g)}$$

for $g \in G$. The action of $g \in G$ on q as in (2.8) is given by multiplication by $\prod_{i=1}^r (1 + x_i)^{\chi_i(g)}$. That is, we have the formula

$$g \cdot q = \sum_{\|k\| < n} \left(\sum_{0 \leq k'_i \leq k_i} \binom{\chi(g)}{k'} g^{\alpha_{k-k'}} \right) x^k, \quad (2.9)$$

where the second sum is over r -tuples k' of nonnegative integers with $k'_i \leq k_i$ for each i , and where we have set $\binom{\chi(g)}{k'} = \binom{\chi_1(g)}{k'_1} \cdots \binom{\chi_r(g)}{k'_r}$.

Note that our assumption on the cardinality of R can be rephrased as saying that either $c = 0$ or R is a quotient of A_1 such that $|R| < \frac{p}{n}|A_1|$. With our notation and this assumption established, we can give an explicit formula for $\Psi^{(n)}$.

Proposition 2.3.2. *Let $f: G \rightarrow T \otimes_R \Omega/I^n$ be a 1-cocycle, and write*

$$f = \sum_{\|k\| < n} \lambda_k x^k$$

with $\lambda_k: G \rightarrow T$. Then $\Psi^{(n)}$ takes the class of f to the class of the 2-cocycle

$$(g, h) \mapsto \sum_{\|k\|=n} \left(\sum_{0 < k'_i \leq k_i} \binom{\chi(g)}{k'} g^{\lambda_{k-k'}(h)} \right) x^k,$$

where the first sum is taken over r -tuples $k = (k_1, \dots, k_r)$ of nonnegative integers summing to n , and the second sum is taken over nonzero r -tuples k' of nonnegative integers with $k'_i \leq k_i$ for all i .

Proof. Consider the set-theoretic section

$$s_n: T \otimes_R \Omega/I^n \rightarrow T \otimes_R \Omega/I^{n+1} \quad (2.10)$$

that takes a sum as in (2.8) to the same expression in the larger module. Let $\tilde{f} = s_n \circ f$. By definition $\Psi^{(n)}(f)$ is the class of $d\tilde{f}$, where

$$d\tilde{f}(g, h) = \tilde{f}(g) + g\tilde{f}(h) - \tilde{f}(gh)$$

for $g, h \in G$. Since f is a cocycle, the right-hand side of this expression is equal to the degree n part of $g\tilde{f}(h)$, which by (2.9) is exactly as in the statement of the proposition. \square

For general H , pro- p but not necessarily abelian, we can use this computation to see that $\Psi^{(1)}$ is given by cup products. We consider the case that H is a quotient of G such that the abelianization H^{ab} of H is finitely generated and pro- p . As before, but now for H^{ab} in place of H , there are nonnegative integers $r \geq c$ and positive integers $t_1 \leq \dots \leq t_c$ such that

$$H^{\text{ab}} \cong \bigoplus_{i=1}^r A_i, \quad (2.11)$$

where $A_i = \mathbb{Z}/p^{t_i}\mathbb{Z}$ for $i = 1, \dots, c$ and $A_i = \mathbb{Z}_p$ for $i = c + 1, \dots, r$. For $i = 1, \dots, r$, we let $\chi_i: G \rightarrow A_i$ denote the quotient map $G \rightarrow A_i$. We take $n = 1$, and our condition on the cardinality of R becomes $s \leq t_1$ when $c \geq 1$.

Fix generators h_1, \dots, h_r of H such that each h_i maps to $1 \in A_i$ under the composition of the quotient map and the isomorphism in (2.11). There is an isomorphism $I/I^2 \cong H^{\text{ab}} \otimes_{\mathbb{Z}_p} R$ taking the image of $x_i = [h_i] - 1$ to $h_i \otimes 1$.

Proposition 2.3.3. *Let H be a finitely generated pro- p group with H^{ab} as in (2.11), let I be the augmentation ideal in $\Omega = R[[H]]$, and let χ_i and x_i for $1 \leq i \leq r$ be as in the previous paragraph. For any 1-cocycle $f: G \rightarrow T$, we have*

$$\Psi^{(1)}(f) = \sum_{i=1}^r (\chi_i \cup f)x_i \in H^2(G, T) \otimes_R I/I^2.$$

Proof. Let $\Omega' = R[[H^{\text{ab}}]]$ with augmentation ideal $I' \subset \Omega'$. Both Ω/I and Ω'/I' are identified with R via the augmentation maps, and there are also compatible isomorphisms between the graded quotients I/I^2 and $I'/(I')^2$ and the R -module $H^{\text{ab}} \otimes_{\mathbb{Z}_p} R$. It follows that the canonical map $\Omega \rightarrow \Omega'$ induces an isomorphism $\Omega/I^2 \cong \Omega'/(I')^2$. Thus $\Psi^{(1)}$ equals the first generalized Bockstein map for H^{ab} , and the proposition follows from the case $n = 1$ of Proposition 2.3.2. \square

This result was previously studied by the third author in the context of Iwasawa theory, where these maps are referred to as reciprocity maps with restricted ramification: see for instance [Sh2, Lemma 4.1] for its introduction. In the following section, we study analogous results for $\Psi^{(n)}$ with $n > 1$ in terms of higher Massey products.

3 Massey products

The notion of Massey products that we will use is conveniently stated using the theory of generalized matrix algebras, as found in [BeCh]. We require only a simple upper-triangular version of these algebras.

3.1 Upper-triangular generalized matrix algebras

Let n be a positive integer, and let R be a commutative ring.

Definition 3.1.1. An n -dimensional *upper-triangular generalized matrix algebra* \mathcal{A} over R (or, R -UGMA) is an R -algebra formed out of the data of

- finitely generated R -modules $A_{i,j}$ for $1 \leq i \leq j \leq n$ with $A_{i,j} = R$ if $i = j$, and
- R -module homomorphisms $\varphi_{i,j,k}: A_{i,j} \otimes_R A_{j,k} \rightarrow A_{i,k}$ for all $1 \leq i \leq j \leq k \leq n$ which are induced by the given R -actions if $i = j$ or $j = k$

such that the two resulting maps

$$A_{i,j} \otimes_R A_{j,k} \otimes_R A_{k,l} \rightarrow A_{i,l}$$

coincide for all $1 \leq i < j < k < l \leq n$. The tuple $(A_{i,j}, \varphi_{i,j,k})$ defines an R -algebra \mathcal{A} with underlying R -module

$$\mathcal{A} = \bigoplus_{1 \leq i \leq j \leq n} A_{i,j},$$

and multiplication given by matrix multiplication: that is, for $a = (a_{i,j})$ and $b = (b_{i,j})$ in \mathcal{A} , the (i, j) -entry $(ab)_{i,j}$ of ab is

$$(ab)_{i,j} = \sum_{k=i}^j \varphi_{i,k,j}(a_{i,k} \otimes b_{k,j}).$$

Our interest is in the multiplicative group $\mathcal{U} = \mathcal{U}(\mathcal{A})$ of *unipotent* matrices in a UGMA \mathcal{A} , i.e., those $a = (a_{i,j})$ with $a_{i,i} = 1$ for all i . We shall often take the quotient $\mathcal{U}' = \mathcal{U}'(\mathcal{A})$ of this \mathcal{U} by its central subgroup $\mathcal{Z} = \mathcal{Z}(\mathcal{A})$ of *unipotent central* elements, i.e., those $a \in \mathcal{U}$ with $a_{i,j} = 0$ for all $(i, j) \neq (1, n)$.

The following is the key example for our purposes.

Example 3.1.2. Let M be a finitely generated R -module, and let m be a positive integer less than n . We define an n -dimensional R -UGMA $\mathcal{A}_n(M, m)$ as follows. Set

$$A_{i,j} = \begin{cases} M & \text{if } i \leq m < j, \\ R & \text{otherwise} \end{cases}$$

and take the maps $\varphi_{i,j,k}$ to be the R -module structure maps. This makes sense since, given $i \leq j \leq k$, at least one of $A_{i,j}$ and $A_{j,k}$ must be R , as m cannot satisfy both $m < j$ and $j \leq m$.

Let us write $\mathcal{U}_n(M, m)$ for $\mathcal{U}(\mathcal{A}_n(M, m))$ and $\mathcal{U}'_n(M, m)$ for $\mathcal{U}'(\mathcal{A}_n(M, m))$. To make this easier to visualize, note that we can write $\mathcal{U}_n(M, m)$ in “block matrix” form as

$$\mathcal{U}_n(M, m) = \begin{pmatrix} U_m(R) & M_{m,n-m}(M) \\ 0 & U_{n-m}(R) \end{pmatrix},$$

where $U_k(R) \leq GL_k(R)$ denotes the group of upper-triangular unipotent matrices and $M_{k,l}(M)$ denotes the additive group of k -by- l matrices with entries in M for positive integers k and l . The latter group is endowed with a left $U_k(R)$ -action and a commuting right $U_l(R)$ -action. Put differently, $\mathcal{A}_n(M, m)$ itself is a sort of 2-by-2 generalized matrix algebra, allowing noncommutative rings on the diagonal and bimodules in the non-diagonal entries.

We actually need to use *profinite UGMAs* defined just as in Definition 3.1.1 using profinite rings R and compact R -modules $A_{i,j}$, but now assuming that the induced multiplication maps $A_{i,j} \times A_{j,k} \rightarrow A_{i,k}$ are continuous. Alternatively, the maps $\varphi_{i,j,k}$ can be replaced by maps of completed tensor products over R in the definition.

Though unnecessary, to keep things simple, let us suppose that the compact R -modules $A_{i,j}$ in a profinite R -UGMA are R -finitely generated. This forces them to have the adic topology for any directed system of ideals that are open neighborhoods of zero. Moreover, their tensor products and completed tensor products are then abstractly isomorphic, and so we may in particular view the tensor products $A_{i,j} \otimes_R A_{j,k}$ themselves as compact R -modules. (For a slightly longer discussion of this, see [LiSh, Section 2.3].)

Note that any profinite R -UGMA \mathcal{A} has a topology as a finite direct product of the compact R -modules $A_{i,j}$, and \mathcal{U} inherits the subspace topology.

We also want to make a second modification, allowing a continuous action of G .

Definition 3.1.3. For a profinite ring R and a profinite group G , a *profinite (R, G) -UGMA* is the data of a profinite R -UGMA \mathcal{A} together with a continuous G -action on each $A_{i,j}$ such that

- the action on $A_{i,i} = R$ is trivial for all i , and

- the maps $\varphi_{i,j,k}$ are maps of $R[[G]]$ -modules, where $A_{i,j} \otimes_R A_{j,k}$ is given the diagonal action of G .

We remark that, aside from issues of finite generation, the difference between a profinite $R[[G]]$ -UGMA and a profinite (R, G) -UGMA is that in the former, each $A_{i,i} = R[[G]]$, whereas in the latter, each $A_{i,i}$ is R with the trivial G -action. We are interested in the latter structure.

Example 3.1.4. If R is a profinite ring and T is a compact $R[[G]]$ -module (that is R -finitely generated), then the R -UGMA $\mathcal{A}_n(T, m)$ of Example 3.1.2 has a natural structure of a profinite (R, G) -UGMA by letting G act on $A_{i,j}$ via its action on T if $i \leq m < j$ and trivially otherwise.

3.2 Defining systems and Massey products

Let R be a profinite ring, let G be a profinite group, and let $n \geq 2$. Let T_1, \dots, T_n be compact $R[[G]]$ -modules that are R -finitely generated for simplicity, and let $\chi_i: G \rightarrow T_i$ be continuous 1-cocycles for $1 \leq i \leq n$. In this section, we define *Massey products* of these cocycles, which will be 2-cocycles that depend on a number of choices constituting a *defining system*.

Definition 3.2.1. A *defining system* for the Massey product of χ_1, \dots, χ_n is the data of

- an $(n + 1)$ -dimensional profinite (R, G) -UGMA \mathcal{A} and
- a (nonabelian) continuous 1-cocycle $\rho: G \rightarrow \mathcal{U}'$

such that $A_{i,i+1} = T_i$ for $1 \leq i \leq n$, and the composition of ρ with projection to $A_{i,i+1}$ is χ_i .

Given a defining system $\rho: G \rightarrow \mathcal{U}'$, there is a unique function $\tilde{\rho}: G \rightarrow \mathcal{U}$ lifting ρ and having zero as the $(1, n + 1)$ -entry of $\tilde{\rho}(g)$ for all $g \in G$. We let $\rho_{i,j}: G \rightarrow A_{i,j}$ be the map given by taking the (i, j) -entry of $\tilde{\rho}$.

Definition 3.2.2. Given a defining system ρ , the *n -fold Massey product* $(\chi_1, \dots, \chi_n)_\rho \in H^2(G, A_{1,n+1})$ is the class of the 2-cocycle

$$(g, h) \mapsto \sum_{i=2}^n \varphi_{1,i,n+1}(\rho_{1,i}(g) \otimes g\rho_{i,n+1}(h))$$

that sends (g, h) to the $(1, n + 1)$ -entry of $\tilde{\rho}(g) \cdot g\tilde{\rho}(h)$.

In the remainder of the paper, we will restrict our attention to the setting of the $(n + 1)$ -dimensional profinite (R, G) -UGMAs of the form $\mathcal{A}_{n+1}(T, m)$ defined in Examples 3.1.2 and 3.1.4. This means in particular that we only consider n -fold Massey products for which there

is an m with $1 \leq m \leq n$ such that $T_m = T$ and $T_i = R$ for $i \neq m$. In particular, we will always have $(\chi_1, \dots, \chi_n)_\rho \in H^2(G, T)$.

In [Sh1], the third author considered the case in which $m = n$ and $\chi_1 = \dots = \chi_{n-1}$ in a Galois-cohomological setting. In that case, the key idea for relating Massey products to graded pieces of Iwasawa cohomology groups was to consider only a restricted set of defining systems referred to as *proper defining systems*. We will consider a more general notion of proper defining system that depends on extra data we call a *partial defining system*. In [Sh1], the partial defining system comes from unipotent binomial matrices, which we review in Section 4.2 below.

3.3 Massey products relative to proper defining systems

Fix an integer $n \geq 2$ and two integers $a, b \geq 0$ with $a + b = n$. Let $Z^i(G, M)$ for a profinite $R[[G]]$ -module M denote the group of continuous i -cocycles on G valued in M . Choose tuples

$$\alpha = (\alpha_1, \dots, \alpha_a) \in Z^1(G, R)^a \quad \text{and} \quad \beta = (\beta_1, \dots, \beta_b) \in Z^1(G, R)^b$$

and a compact $R[[G]]$ -module T that is finitely generated as an R -module.

We next consider a pair of homomorphisms that constitute a part of the defining systems for $(n + 1)$ -fold Massey products $(\alpha_1, \dots, \alpha_a, \lambda, \beta_1, \dots, \beta_b)$, where $\lambda \in Z^1(G, T)$ is allowed to vary. We write the collection of such Massey products as (α, \cdot, β) for short.

Definition 3.3.1. A *partial defining system* for $(n + 1)$ -fold Massey products (α, \cdot, β) is a pair of homomorphisms

$$\phi: G \rightarrow U_{a+1}(R) \quad \text{and} \quad \theta: G \rightarrow U_{b+1}(R)$$

such that α is the off-diagonal of ϕ and β is the off-diagonal of θ , i.e., $\phi_{i,i+1} = \alpha_i$ for $1 \leq i \leq a$ and $\theta_{i,i+1} = \beta_i$ for $1 \leq i \leq b$.

More specifically, an (a, b) -*partial defining system* is a partial defining system restricting to some pair $(\alpha, \beta) \in Z^1(G, R)^a \times Z^1(G, R)^b$.

Recall that

$$\mathcal{U}_{n+2}(T, a + 1) = \begin{pmatrix} U_{a+1}(R) & M_{a+1,b+1}(T) \\ & U_{b+1}(R) \end{pmatrix}.$$

We may then write the quotient by the unipotent central matrices as

$$\mathcal{U}'_{n+2}(T, a + 1) = \begin{pmatrix} U_{a+1}(R) & M'_{a+1,b+1}(T) \\ & U_{b+1}(R) \end{pmatrix}$$

for $M'_{a+1,b+1}(T) = M_{a+1,b+1}(T)/T$, where T is identified with the matrices that are zero outside the $(1, b + 1)$ -entry.

Definition 3.3.2. Given a 1-cocycle $\lambda: G \rightarrow T$, a *proper defining system* for an $(n + 1)$ -fold Massey product (α, λ, β) relative to a *partial defining system* (ϕ, θ) is a continuous 1-cocycle

$$\rho: G \rightarrow \mathcal{U}'_{n+2}(T, a + 1)$$

of the form

$$\rho = \begin{pmatrix} \phi & \kappa \\ 0 & \theta \end{pmatrix}$$

for some $\kappa: G \rightarrow M'_{a+1,b+1}(T)$ with $\kappa_{a+1,1} = \lambda$.

The advantage of proper defining systems is that they are parameterized by abelian, rather than nonabelian, cocycles. To show this, we introduce a compact $R[[G]]$ -module $\mathfrak{U}_{\phi,\theta}(T)$ such that proper defining systems in T relative to (ϕ, θ) correspond to 1-cocycles with values in $\mathfrak{U}_{\phi,\theta}(T)$.

Consider the compact R -module $\mathfrak{U}_{n+2}(R)$ that is the R -module of strictly upper-triangular $(n + 2)$ -dimensional square matrices. The group $U_{n+2}(R)$ acts continuously on $\mathfrak{U}_{n+2}(R)$ by conjugation. We consider a $R[[U_{n+2}(R)]]$ -submodule $\mathfrak{U}_{a,b}(R)$ of $\mathfrak{U}_{n+2}(R)$ given by

$$\{x = (x_{ij}) \in M_{n+2}(R) \mid x_{ij} = 0 \text{ if } j \leq a + 1 \text{ or } i \geq a + 2\}.$$

In other words, breaking $M_{n+2}(R)$ into blocks using the partition $n + 2 = (a + 1) + (b + 1)$ and using block-matrix notation, we have

$$\mathfrak{U}_{a,b}(R) = \begin{pmatrix} 0 & M_{a+1,b+1}(R) \\ 0 & 0 \end{pmatrix}.$$

Given a partial defining system (ϕ, θ) , we consider $\mathfrak{U}_{a,b}(R)$ as a G -module via the continuous homomorphism

$$G \rightarrow U_{n+2}(R), \quad g \mapsto \begin{pmatrix} \phi(g) & 0 \\ 0 & \theta(g) \end{pmatrix}.$$

We define an $R[[G]]$ -module $\mathfrak{U}_{\phi,\theta}(T)$ as $\mathfrak{U}_{a,b}(R) \otimes_R T$ with the diagonal G -action. We also have the following equivalent definition, which has the benefit of being more explicit:

- $\mathfrak{U}_{\phi,\theta}(T) = M_{a+1,b+1}(T)$ as an R -module,

- the action map $G \rightarrow \text{End}(M_{a+1,b+1}(T))$ is given, for $g \in G$ and $x \in M_{a+1,b+1}(T)$, by

$$g \star x = \phi(g) \cdot gx \cdot \theta(g)^{-1}.$$

where gx means apply the g action on T to each matrix entry, and the multiplication denoted by “ \cdot ” is of matrices.

Going forward, we use the latter description of $\mathfrak{U}_{\phi,\theta}(T)$, so consider it as consisting of $(a+1)$ -by- $(b+1)$ matrices, rather than as a subgroup of $M_{n+2}(R)$. Note that $\mathfrak{U}_{\phi,\theta}(T)$ contains a copy of T as an $R[[G]]$ -submodule by inclusion in the $(1, b+1)$ -entry. Let

$$\mathfrak{U}'_{\phi,\theta}(T) = \mathfrak{U}_{\phi,\theta}(T)/T.$$

Let $x \mapsto \tilde{x}$ denote the R -module section $\mathfrak{U}'_{\phi,\theta}(T) \rightarrow \mathfrak{U}_{\phi,\theta}(T)$ given by filling in the $(1, b+1)$ -entry as 0.

Lemma 3.3.3. *Let (ϕ, θ) be a partial defining system for Massey products (α, \cdot, β) . Then the map that takes a continuous 1-cocycle $\kappa': G \rightarrow \mathfrak{U}'_{\phi,\theta}(T)$ to a map $\rho: G \rightarrow \mathfrak{U}'_{n+2}(T, a)$ given by*

$$\rho = \begin{pmatrix} \phi & \kappa'\theta \\ 0 & \theta \end{pmatrix}.$$

is a bijection between $Z^1(G, \mathfrak{U}'_{\phi,\theta}(T))$ and the set of proper defining systems in T relative to (ϕ, θ) .

Proof. Given a cochain $\kappa': G \rightarrow \mathfrak{U}'_{\phi,\theta}(T)$, set $\kappa = \kappa'\theta: G \rightarrow \mathfrak{U}'_{\phi,\theta}(T)$. We have to check that

$$\rho = \begin{pmatrix} \phi & \kappa \\ 0 & \theta \end{pmatrix}$$

is a cocycle if and only if κ' is a cocycle. Matrix multiplication tells us that ρ is a cocycle if and only if

$$\kappa(gh) = \phi(g)g\kappa(h) + \kappa(g)\theta(h). \quad (3.1)$$

The cochain κ' is a cocycle if and only if the second equality holds in the following string of equalities

$$\begin{aligned} \kappa(gh) &= \kappa'(gh)\theta(gh) \\ &= (g \star \kappa'(h) + \kappa'(g))\theta(gh) \\ &= (\phi(g)g\kappa'(h)\theta(g)^{-1} + \kappa'(g))\theta(g)\theta(h) \\ &= \phi(g)g\kappa'(h)\theta(h) + \kappa'(g)\theta(g)\theta(h) \\ &= \phi(g)g\kappa(h) + \kappa(g)\theta(h), \end{aligned}$$

hence the result. □

The value of the Massey product associated to a proper defining system is also a value of a connecting homomorphism for an exact sequence attached to the underlying partial defining system.

Theorem 3.3.4. *Let (ϕ, θ) be a partial defining system for (α, \cdot, β) . Let $\kappa' \in Z^1(G, \mathfrak{U}'_{\phi, \theta}(T))$, and let $\rho = \begin{pmatrix} \phi & \kappa'\theta \\ 0 & \theta \end{pmatrix}$ be the associated proper defining system as in Lemma 3.3.3. Consider the short exact sequence*

$$0 \rightarrow T \rightarrow \mathfrak{U}_{\phi, \theta}(T) \rightarrow \mathfrak{U}'_{\phi, \theta}(T) \rightarrow 0.$$

Then the image of the class of κ' under the connecting map

$$\partial: H^1(G, \mathfrak{U}'_{\phi, \theta}(T)) \rightarrow H^2(G, T)$$

is the $(n+1)$ -fold Massey product $(\alpha_1, \dots, \alpha_a, \kappa'_{a+1,1}, \beta_1, \dots, \beta_b)_\rho$.

Proof. Let $\kappa = \kappa'\theta: G \rightarrow \mathfrak{U}'_{\phi, \theta}(T)$, and let $\tilde{\kappa}$ be its unique lift to $\mathfrak{U}_{\phi, \theta}(T)$ with $\tilde{\kappa}(g)$ having zero $(1, b+1)$ -entry for all $g \in G$. The map $\tilde{\kappa}' = \tilde{\kappa}\theta^{-1}: G \rightarrow \mathfrak{U}'_{\phi, \theta}(T)$ is then a lift of κ' . By definition, the image of κ is represented by the 2-cocycle that is given by taking the $(1, b+1)$ -entry of $d\tilde{\kappa}'$. We have

$$\begin{aligned} d\tilde{\kappa}'(g, h) &= \tilde{\kappa}'(g) + g \star \tilde{\kappa}'(h) - \tilde{\kappa}'(gh) \\ &= \tilde{\kappa}(g)\theta(g)^{-1} + \phi(g)g\tilde{\kappa}(h)\theta(h)^{-1}\theta(g)^{-1} - \tilde{\kappa}(gh)\theta(gh)^{-1} \\ &= (\tilde{\kappa}(g)\theta(h) + \phi(g)g\tilde{\kappa}(h) - \tilde{\kappa}(gh))\theta(gh)^{-1}. \end{aligned}$$

Since κ satisfies (3.1), we have $\tilde{\kappa}(g)\theta(h) + \phi(g)g\tilde{\kappa}(h) - \tilde{\kappa}(gh) \in T$, and T is fixed under the action of right multiplication by an element of $U_{b+1}(R)$. Since $\tilde{\kappa}(gh)$ has zero $(1, b+1)$ -entry, the $(1, b+1)$ -entries of $d\tilde{\kappa}'(g, h)$ and $\tilde{\kappa}(g)\theta(h) + \phi(g)g\tilde{\kappa}(h)$ are equal.

The Massey product $(\alpha_1, \dots, \alpha_a, \kappa_{a+1,1}, \beta_1, \dots, \beta_b)_\rho$ (and note that $\kappa_{a+1,1} = \kappa'_{a+1,1}$) is the $(1, n+2)$ -entry of $\tilde{\rho}(g) \cdot g\tilde{\rho}(h)$, where

$$\tilde{\rho} = \begin{pmatrix} \phi & \tilde{\kappa} \\ 0 & \theta \end{pmatrix}.$$

The result then follows from the fact that

$$\tilde{\rho}(g) \cdot g\tilde{\rho}(h) = \begin{pmatrix} \phi(g) & \tilde{\kappa}(g) \\ 0 & \theta(g) \end{pmatrix} \begin{pmatrix} \phi(h) & g\tilde{\kappa}(h) \\ 0 & \theta(h) \end{pmatrix} = \begin{pmatrix} \phi(gh) & \phi(g)g\tilde{\kappa}(h) + \tilde{\kappa}(g)\theta(h) \\ 0 & \theta(gh) \end{pmatrix}.$$

□

In fact, the proof of Theorem 3.3.4 gives an explicit map $Z^1(G, \mathfrak{U}'_{\phi, \theta}(T)) \rightarrow Z^2(G, T)$, taking a 1-cocycle κ' to the $(1, b+1)$ -entry of $d\tilde{\kappa}'$, for the specific lift $\tilde{\kappa}'$ of κ' defined therein.

4 Massey products as values of Bockstein maps

We return to the setting and notation of Section 2. We first discuss a general result that gives partial information about the generalized Bockstein map $\Psi^{(n)}$ in terms of Massey products. Then we discuss specific examples where this information completely determines $\Psi^{(n)}$.

4.1 Partial defining systems and Bockstein maps

Fix integers $a, b \geq 0$ such that $a + b = n$ and group homomorphisms

$$\phi: H \rightarrow U_{a+1}(R) \quad \text{and} \quad \theta: H \rightarrow U_{b+1}(R),$$

so viewing ϕ and θ as maps from G via precomposition with the quotient map, the pair (ϕ, θ) is an (a, b) -partial defining system. We let $\alpha = (\phi_{i,i+1})_i$ and $\beta = (\theta_{i,i+1})_i$, so this partial defining system is of Massey products (α, \cdot, β) . If $b = 0$, we often refer to the pair (ϕ, θ) simply as ϕ .

Lemma 4.1.1. *Let $e \in \mathfrak{U}_{a,b}(R)$ be the matrix with $(a + 1, 1)$ -entry equal to 1 and all other entries 0. There is a continuous $R[[G]]$ -module homomorphism $p_{\phi,\theta}: \Omega/I^{n+1} \rightarrow \mathfrak{U}_{a,b}(R)$ given on the images of group elements by*

$$p_{\phi,\theta}([h]) = \phi(h) \cdot e \cdot \theta(h)^{-1}.$$

The image of I^n is contained in the submodule of matrices that are zero outside of their $(1, b + 1)$ -entries.

Proof. The map $\tilde{p}_{\phi,\theta}: \Omega \rightarrow \mathfrak{U}_{a,b}(R)$ inducing $p_{\phi,\theta}$ is given by the action of H on $e \in \mathfrak{U}_{a,b}(R)$ via the composite homomorphism

$$H \xrightarrow{\rho_{\phi,\theta}} U_{n+2}(R) \xrightarrow{\text{ad}} \text{Aut}(\mathfrak{U}_{a,b}(R)),$$

where $\rho_{\phi,\theta}: H \rightarrow U_{n+2}(R)$ is given by

$$\rho_{\phi,\theta}(h) = \begin{pmatrix} \phi(h) & 0 \\ 0 & \theta(h) \end{pmatrix}$$

and ad denotes the conjugation action. The action of G on Ω is given by the homomorphism $G \rightarrow H$, and the action of G on $\mathfrak{U}_{a,b}(R)$ is given by the composite of this map with $H \rightarrow \text{Aut}(\mathfrak{U}_{a,b}(R))$, so $\tilde{p}_{\phi,\theta}$ is G -equivariant. We must show it factors through Ω/I^{n+1} .

Let $J \subset R[[U_{n+2}(R)]]$ be the augmentation ideal. Since the H -action factors through $U_{n+2}(R)$, we have $I^k \mathfrak{U}_{a,b}(R) \subseteq J^k \mathfrak{U}_{a,b}(R)$ for all k . It is easy to see inductively that

$$J^k \mathfrak{U}_{n+2}(R) = \{(a_{ij}) \in M_{n+2}(R) \mid a_{ij} = 0 \text{ if } j - i \leq k\}.$$

In particular, $J^{n+1}\mathfrak{U}_{n+2}(R) = 0$ and

$$J^n\mathfrak{U}_{n+2}(R) = \{(a_{ij}) \in M_{n+2}(R) \mid a_{ij} = 0 \text{ if } (i, j) \neq (1, n+2)\}.$$

Still viewing $\mathfrak{U}_{a,b}(R)$ as a subgroup of $\mathfrak{U}_{n+2}(R)$, the containments

$$I^k\mathfrak{U}_{a,b}(R) \subseteq J^k\mathfrak{U}_{a,b}(R) \subseteq J^k\mathfrak{U}_{n+2}(R)$$

imply the result. □

Lemma 4.1.1 implies that there is a map of short exact sequences of $R[[G]]$ -modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & T \otimes_R I^n/I^{n+1} & \longrightarrow & T \otimes_R \Omega/I^{n+1} & \longrightarrow & T \otimes_R \Omega/I^n & \longrightarrow & 0 \\ & & \downarrow p_{\phi,\theta} & & \downarrow p_{\phi,\theta} & & \downarrow p_{\phi,\theta} & & \\ 0 & \longrightarrow & T & \longrightarrow & \mathfrak{U}_{\phi,\theta}(T) & \longrightarrow & \mathfrak{U}'_{\phi,\theta}(T) & \longrightarrow & 0, \end{array} \quad (4.1)$$

where $p_{\phi,\theta}$ is the tensor product with T of the map in Lemma 4.1.1 coming from $\rho_{\phi,\theta}$. As a direct consequence of this commutativity and Theorem 3.3.4, we have the following.

Theorem 4.1.2. *Let $\phi: G \rightarrow U_{a+1}(R)$ and $\theta: G \rightarrow U_{b+1}(R)$ restrict to $\alpha \in Z^1(G, R)^a$ and $\beta \in Z^1(G, R)^b$ as above. Let $f \in Z^1(G, T \otimes_R \Omega/I^n)$, and let ρ denote the proper defining system relative to (ϕ, θ) associated to $p_{\phi,\theta} \circ f$ by Lemma 3.3.3. Then we have*

$$p_{\phi,\theta}(\Psi^{(n)}([f])) = (\alpha, (p_{\phi,\theta} \circ f)_{a+1,1}, \beta)_\rho$$

in $H^2(G, T)$. Here, the maps $p_{\phi,\theta}$ on the left and right are those induced on cohomology by the the left and right vertical maps in (4.1).

We will give examples of groups H and integers n such that there is a set X of choices of (ϕ, θ) for which the map

$$H^2(G, T) \otimes_R I^n/I^{n+1} \xrightarrow{\prod_{(\phi,\theta) \in X} p_{\phi,\theta}} \prod_{(\phi,\theta) \in X} H^2(G, T)$$

is injective. In such cases, Theorem 4.1.2 shows that the generalized Bockstein map $\Psi^{(n)}$ is determined by Massey products. In the rest of this section, we consider some specific examples in detail.

4.2 Unipotent binomial matrices

We introduce the *unipotent binomial matrices*, which are a source of many partial defining systems. Let n denote a positive integer, and let p be a prime number.

Let u_n denote the $(n + 1)$ -dimensional nilpotent upper triangular matrix

$$u_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \vdots \\ & & 0 & \ddots & 0 \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}.$$

For any $k \geq 1$, the matrix u_n^k has (i, j) -entry 1 if $j - i = k$ and 0 otherwise. In particular, we have $u_n^{n+1} = 0$.

Let $\begin{bmatrix} \cdot \\ n \end{bmatrix}: \mathbb{Z}_p \rightarrow U_{n+1}(\mathbb{Z}_p)$ denote the unique continuous homomorphism to $(n + 1)$ -dimensional unipotent matrices with \mathbb{Z}_p -entries such that $\begin{bmatrix} 1 \\ n \end{bmatrix} = 1 + u_n$. By the binomial theorem, for $a \in \mathbb{Z}$ we have

$$\begin{bmatrix} a \\ n \end{bmatrix} = (1 + u_n)^a = \sum_{k=0}^n \binom{a}{k} u_n^k = \begin{pmatrix} 1 & a & \binom{a}{2} & \cdots & \binom{a}{n} \\ & 1 & a & \ddots & \vdots \\ & & 1 & \ddots & \binom{a}{2} \\ & & & \ddots & a \\ & & & & 1 \end{pmatrix}.$$

If $t \geq s$ and $n < p^{t-s+1}$, then the composite map

$$\mathbb{Z} \xrightarrow{\begin{bmatrix} \cdot \\ n \end{bmatrix}} U_{n+1}(\mathbb{Z}_p) \rightarrow U_{n+1}(\mathbb{Z}/p^s\mathbb{Z})$$

that sends a to $(1 + u_n)^a$ modulo p^s factors through $\mathbb{Z}/p^t\mathbb{Z}$ by Lemma 2.3.1 applied with $x = u_n$. By abuse of notation, we again denote the resulting map $\mathbb{Z}/p^t\mathbb{Z} \rightarrow U_{n+1}(\mathbb{Z}/p^s\mathbb{Z})$ by $\begin{bmatrix} \cdot \\ n \end{bmatrix}$. In particular, the map $\binom{\cdot}{n}: \mathbb{Z} \rightarrow \mathbb{Z}/p^s\mathbb{Z}$ given by $a \mapsto \binom{a}{n} \pmod{p^s}$ factors through $\mathbb{Z}/p^t\mathbb{Z}$, and we abuse notation to also denote the resulting map $\mathbb{Z}/p^t\mathbb{Z} \rightarrow \mathbb{Z}/p^s\mathbb{Z}$ by $\binom{\cdot}{n}$.

The following lemma, phrased conveniently for our purposes, summarizes the above discussion.

Lemma 4.2.1. *Let A be a quotient of the ring \mathbb{Z}_p and R be a quotient of A . Let H be a profinite group and $\chi: H \rightarrow A$ be a continuous homomorphism. Suppose that either $A = \mathbb{Z}_p$ or $|R| < \frac{p}{n}|A|$. Then there is a homomorphism*

$$\begin{bmatrix} \chi \\ n \end{bmatrix}: H \rightarrow U_{n+1}(R),$$

defined by $[\begin{smallmatrix} \chi \\ n \end{smallmatrix}](h) = [\begin{smallmatrix} \chi^{(h)} \\ n \end{smallmatrix}]$ for all $h \in H$.

Proof. If $|R| = p^s$ and $|A| = p^t$, then $|R| < \frac{p}{n}|A|$ if and only if $n < p^{t-s+1}$. \square

4.3 Procyclic case

In this subsection, we fix a surjective homomorphism $\chi: G \rightarrow A$, where A is a nonzero quotient of \mathbb{Z}_p . We suppose that our ring R is a nonzero quotient of A with $A = \mathbb{Z}_p$ or $n|R| < p|A|$. We define H to be the coimage of χ , so $H \cong A$. We fix $h \in H$ to be the preimage of $1 \in A$ and let $x = [h] - 1 \in \Omega$, which is a generator of the augmentation ideal I . Our assumption on the size of R implies that $\Omega/I^j \cong R[x]/(x^j)$ for all $j \leq n+1$ by the discussion of Section 2.3. In particular, we have $I^n/I^{n+1} = Rx^n$.

The $(n, 0)$ -proper defining systems relative to $\phi = [\begin{smallmatrix} \chi \\ n \end{smallmatrix}]$ and the trivial map θ to $U_1(R) = \{1\}$ agree with the proper defining systems considered in [Sh1] for Galois groups. We give an interpretation of the resulting Massey products in terms of generalized Bockstein maps. That is, let us apply the discussion of Section 4.1 to this situation. We have $\alpha = (\chi, \dots, \chi) \in Z^1(G, R)^n$, which we denote by $\chi^{(n)}$. We denote $\mathfrak{U}_{\phi, \theta}(T)$ by $\mathfrak{U}_{[\begin{smallmatrix} \chi \\ n \end{smallmatrix}]}(T)$.

Set $p_n = p_{[\begin{smallmatrix} \chi \\ n \end{smallmatrix}], 0}$ for brevity. Then the diagram (4.1) becomes

$$\begin{array}{ccccccc} 0 & \longrightarrow & T \cdot x^n & \longrightarrow & T \otimes_R \Omega/I^{n+1} & \longrightarrow & T \otimes_R \Omega/I^n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow p_n & & \downarrow p_n & & \\ 0 & \longrightarrow & T & \longrightarrow & \mathfrak{U}_{[\begin{smallmatrix} \chi \\ n \end{smallmatrix}]}(T) & \longrightarrow & \mathfrak{U}'_{[\begin{smallmatrix} \chi \\ n \end{smallmatrix}]}(T) & \longrightarrow & 0, \end{array}$$

where p_n is the map attached to $([\begin{smallmatrix} \chi \\ n \end{smallmatrix}], 0)$ by Lemma 4.1.1. Explicitly, the vector $p_n(\sum_{k=0}^n a_k x^k)$ in $M_{n+1,1}(T)$ has i th entry a_{n+1-i} . (See the more general case proven in Lemma 4.4.1 of the next subsection.)

By Lemma 3.3.3, it follows that the proper defining system ρ_{x^n} relative to $[\begin{smallmatrix} \chi \\ n \end{smallmatrix}]$ that is attached to $p_n \circ f$, where

$$f = \sum_{k=0}^{n-1} \lambda_k x^k \in Z^1(G, T \otimes_R \Omega/I^n),$$

satisfies $(\rho_{x^n})_{n+1-k, n+2} = \lambda_k$ for $0 \leq k \leq n-1$. In particular, the element $\lambda = \lambda_0 = (\rho_{x^n})_{n+1, n+2}$ is the image of f under the map

$$Z^1(G, T \otimes_R \Omega/I^n) \rightarrow Z^1(G, T)$$

induced by the augmentation $\Omega/I^n \mapsto \Omega/I = R$.

Theorem 4.1.2 then gives us an explicit description of the values of the generalized Bockstein homomorphism on classes in $H^1(G, T \otimes_R \Omega/I^n)$ as Massey products $(\chi^{(n)}, \cdot)$ relative to $\begin{bmatrix} \chi \\ n \end{bmatrix}$, as follows.

Theorem 4.3.1. *For $f \in Z^1(G, T \otimes_R \Omega/I^n)$, we have*

$$\Psi^{(n)}([f]) = (\chi^{(n)}, \lambda)_{\rho_{x^n}} \cdot x^n,$$

where ρ_{x^n} is the proper defining system relative to $\begin{bmatrix} \chi \\ n \end{bmatrix}$ attached to f , and λ is the image of f in $Z^1(G, T)$.

In particular, we have the following description of the image of $\Psi^{(n)}$.

Corollary 4.3.2. *The image of the generalized Bockstein map $\Psi^{(n)}$ is the set of all $(\chi^{(n)}, \lambda)_{\rho} \cdot x^n$ for Massey products of n copies of χ with 1-cocycles $\lambda \in Z^1(G, T)$ for proper defining systems ρ relative to $\begin{bmatrix} \chi \\ n \end{bmatrix}$ with $\rho_{n+1, n+2} = \lambda$.*

Theorem 2.2.4 provides the following application to the graded quotients of Iwasawa cohomology groups of $N = \ker(\chi: G \rightarrow H)$.

Corollary 4.3.3. *Suppose that G is p -cohomologically finite of p -cohomological dimension 2. Let $P_n(H)$ denote the subgroup of $H^2(G, T) \otimes_R I^n/I^{n+1}$ generated by all $(\chi^{(n)}, \lambda)_{\rho} \cdot x^n$ for proper defining systems ρ relative to $\begin{bmatrix} \chi \\ n \end{bmatrix}$ and $\lambda = \rho_{n+1, n+2}$. We have a canonical isomorphism of R -modules*

$$\frac{I^n H_{\text{Iw}}^2(N, T)}{I^{n+1} H_{\text{Iw}}^2(N, T)} \cong \frac{H^2(G, T) \otimes_R I^n/I^{n+1}}{P_n(H)}.$$

4.4 Pro-bicyclic case

In this subsection, we

- fix a surjective homomorphism $(\chi, \psi): G \twoheadrightarrow A \times B$, where A and B are nonzero quotients of \mathbb{Z}_p ,
- let $a, b \geq 0$ denote integers such that $a + b = n$, and
- suppose that R is a nonzero quotient of both A and B with $a|R| < p|A|$ if A is finite and $b|R| < p|B|$ if B is finite.

Let H be the coimage of (χ, ψ) so that $H \cong A \times B$. Let $h_A, h_B \in H$ be the preimages of $(1, 0), (0, 1) \in A \times B$, respectively, and let $x = [h_A] - 1$ and $y = [h_B] - 1$ so that (x, y) is

the augmentation ideal I of $\Omega = R[[H]]$. We have $\Omega/I^j = R[x, y]/(x, y)^j$ for all $j \leq n$. In particular, we have

$$I^n/I^{n+1} = \bigoplus_{i+j=n} R x^i y^j.$$

We apply the discussion of Section 4.1 to this situation. We take $\phi = \begin{bmatrix} \chi \\ a \end{bmatrix}: H \rightarrow U_a(R)$ and $\theta = \begin{bmatrix} \psi \\ b \end{bmatrix}: H \rightarrow U_b(R)$. We have $\alpha = \chi^{(a)} \in Z^1(G, R)^a$ and $\beta = \psi^{(b)} \in Z^1(G, R)^b$.

Set $p_{a,b} = p_{\begin{bmatrix} \chi \\ a \end{bmatrix}, \begin{bmatrix} \psi \\ b \end{bmatrix}}$ for brevity. In this setting, the diagram (4.1) becomes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{i+j=n} T \cdot x^i y^j & \longrightarrow & T \otimes_R \Omega/I^{n+1} & \longrightarrow & T \otimes_R \Omega/I^n \longrightarrow 0 \\ & & \downarrow & & \downarrow p_{a,b} & & \downarrow p_{a,b} \\ 0 & \longrightarrow & T & \longrightarrow & \mathfrak{U}_{\begin{bmatrix} \chi \\ a \end{bmatrix}, \begin{bmatrix} \psi \\ b \end{bmatrix}}(T) & \longrightarrow & \mathfrak{U}'_{\begin{bmatrix} \chi \\ a \end{bmatrix}, \begin{bmatrix} \psi \\ b \end{bmatrix}}(T) \longrightarrow 0. \end{array} \quad (4.2)$$

Lemma 4.4.1. *The $R[[G]]$ -module map $p_{a,b}: T \otimes_R \Omega/I^{n+1} \rightarrow \mathfrak{U}_{\begin{bmatrix} \chi \\ a \end{bmatrix}, \begin{bmatrix} \psi \\ b \end{bmatrix}}(T)$ is an isomorphism satisfying*

$$p_{a,b} \left(\sum_{k_1+k_2 \leq n} c_{k_1, k_2} x^{k_1} y^{k_2} \right) = (c_{a+1-i, j-1})_{i,j},$$

In particular, the lefthand vertical map in (4.2) is given by projection onto the factor $T \cdot x^a y^b \cong T$.

Proof. This reduces immediately to the case that $T = R$, since we can obtain the case of arbitrary T by R -tensor product with the identity of T . Let e be as in Lemma 4.1.1, the matrix with a single nonzero entry of 1 in the $(a+1, 1)$ -coordinate of $M_{a+1, b+1}(R)$. The (i, j) -entry of $g \star e = \begin{bmatrix} \chi(g) \\ a \end{bmatrix} e \begin{bmatrix} \psi(g) \\ b \end{bmatrix}$ is

$$\begin{pmatrix} \chi(g) \\ a+1-i \end{pmatrix} \begin{pmatrix} \psi(g) \\ j-1 \end{pmatrix}.$$

which agrees with the coefficient of $x^{a+1-i} y^{j-1}$ in $g \cdot 1$ by (2.9). \square

Corollary 4.4.2. *For*

$$f = \sum_{k_1+k_2 < n} \lambda_{k_1, k_2} x^{k_1} y^{k_2} \in Z^1(G, \Omega/I^n \otimes_R T)$$

and $\rho_{x^a y^b}$ the proper defining system relative to $(\begin{bmatrix} \chi \\ a \end{bmatrix}, \begin{bmatrix} \psi \\ b \end{bmatrix})$ attached to $p_{a,b} \circ f$ by Lemma 3.3.3, we have

$$(\rho_{x^a y^b})_{a+1-k_1, a+2+k_2} = \lambda_{k_1, k_2}$$

for all $0 \leq (k_1, k_2) < (a, b)$. In particular, we have $(\rho_{x^a y^b})_{a+1, a+2} = \lambda_{0,0}$, which is the image of f in $Z^1(G, T)$ under the map induced by the quotient $\Omega/I^n \rightarrow \Omega/I = R$.

The following is then a direct consequence of Theorem 4.1.2.

Theorem 4.4.3. *For $f \in Z^1(G, \Omega/I^n \otimes_R T)$, the image of $\Psi^{(n)}([f])$ in*

$$H^2(G, T) \otimes_R I^n/I^{n+1} \cong \bigoplus_{a+b=n} H^2(G, T) \cdot x^a y^b$$

is

$$\sum_{a+b=n} (\chi^{(a)}, \lambda, \psi^{(b)})_{\rho_{x^a y^b}} \cdot x^a y^b,$$

where $\rho_{x^a y^b}$ is the proper defining system relative to $(\begin{bmatrix} \chi \\ a \end{bmatrix}, \begin{bmatrix} \psi \\ b \end{bmatrix})$ attached to $p_{a,b} \circ f$, and λ is the image of f in $Z^1(G, T)$.

Applying Theorem 2.2.4, we obtain the following description of graded quotients of Iwasawa cohomology.

Corollary 4.4.4. *Suppose that G is p -cohomologically finite of p -cohomological dimension 2. Let $P_n(H)$ denote the subgroup of $H^2(G, T) \otimes_R I^n/I^{n+1}$ consisting of all sums $\sum_{a+b=n} (\chi^{(a)}, \lambda, \psi^{(b)})_{\rho_{x^a y^b}} \cdot x^a y^b$, where the $\rho_{x^a y^b}$ and λ are associated to a cocycle in $Z^1(G, \Omega/I^n \otimes_R T)$ as in Proposition 4.4.3. We then have a canonical isomorphism of R -modules*

$$\frac{I^n H_{\text{Iw}}^2(N, T)}{I^{n+1} H_{\text{Iw}}^2(N, T)} \cong \frac{H^2(G, T) \otimes_R I^n/I^{n+1}}{P_n(H)}.$$

4.5 Elementary abelian p -groups

The pattern seen in the cyclic and bicyclic cases does not continue for all finitely generated abelian pro- p groups. To see why, consider the case that $H \cong \mathbb{F}_p^3$ with basis $(\gamma_1, \gamma_2, \gamma_3)$ and dual basis (χ_1, χ_2, χ_3) . For $x_i = [\gamma_i] - 1$, we have a basis of I^n/I^{n+1} consisting of monomials $x_1^{i_1} x_2^{i_2} x_3^{i_3}$ with $i_1 + i_2 + i_3 = n$. Following the pattern of the cyclic and bicyclic cases, one might guess that the coefficient of $x_1^{i_1} x_2^{i_2} x_3^{i_3}$ in $\Psi^{(n)}(f)$ is an $(n+1)$ -fold Massey product involving i_j copies of each χ_j and another cocycle λ determined by f . However, this pattern fails already for $n=3$ and the coefficient of $x_1 x_2 x_3$: any 4-fold Massey product involving the χ_i must have two of these characters beside each other, and thus, to be defined, the cup product of those two characters must vanish. Since these cup products will not vanish in general, we cannot hope for such a general statement to hold.

Nevertheless, at least in some cases, one can still describe the generalized Bockstein maps $\Psi^{(n)}$ in terms of Massey products, at the expense of taking a non-standard basis for I^n/I^{n+1} . In this subsection, we assume that $H \cong \mathbb{F}_p^r$ for some $r \geq 1$. Correspondingly, we take $n < p$ and $R = \mathbb{F}_p$.

We let $V^\vee = \text{Hom}(V, \mathbb{F}_p)$ for an abelian group V . For any element $\chi \in H^\vee$, we have a homomorphism

$$\begin{bmatrix} \chi \\ n \end{bmatrix}: H \rightarrow U_{n+1}(\mathbb{F}_p).$$

Precomposing with $G \rightarrow H$, we may view χ as a character of G . This gives an $(n, 0)$ -partial defining system, and we set $p_{\chi, n} = p_{\begin{bmatrix} \chi \\ n \end{bmatrix}, 0}$ for brevity. By (4.1), the map $p_{\chi, n}$ induces a map $p_{\chi, n}: I^n/I^{n+1} \rightarrow \mathbb{F}_p$, so $p_{\chi, n} \in (I^n/I^{n+1})^\vee$. This defines a function $p_{-, n}: H^\vee \rightarrow (I^n/I^{n+1})^\vee$.

Let us fix an isomorphism $H \xrightarrow{\sim} \mathbb{F}_p^r$, which in turn fixes an ordered dual basis $(\gamma_i)_{i=1}^r$ of H . Setting $x_i = [\gamma_i] - 1 \in \Omega$, this provides an identification

$$\Omega/I^{n+1} = \mathbb{F}_p[x_1, \dots, x_r]/(x_1, \dots, x_r)^{n+1}. \quad (4.3)$$

Then I^n/I^{n+1} has a basis given by $x_1^{d_1} \cdots x_r^{d_r}$ with (d_1, \dots, d_r) ranging over r -tuples of nonnegative integers with $d_1 + \cdots + d_r = n$. We compute $p_{\chi, n}$ on this basis.

Lemma 4.5.1. *Let $\chi \in H^\vee$. For any nonnegative integers d_1, \dots, d_r with sum n , we have*

$$p_{\chi, n}(x_1^{d_1} \cdots x_r^{d_r}) = \prod_{i=1}^r \chi(\gamma_i)^{d_i}.$$

Proof. We have

$$p_{\chi, n}(x_i) = \left(\begin{bmatrix} \chi(\gamma_i) \\ n \end{bmatrix} - 1 \right) e = ((1 + u_n)^{\chi(\gamma_i)} - 1)e \in \mathfrak{U}_{\begin{bmatrix} \chi \\ n \end{bmatrix}},$$

where u_n is as in Section 4.2 and e is as in Lemma 4.1.1. Note that u_n^n has a 1 in its $(1, n+1)$ entry and all other entries 0, and $u_n^{n+1} = 0$. For $d_1 + \cdots + d_r = n$, the value $p_{\chi, n}(x_1^{d_1} \cdots x_r^{d_r})$ is the $(1, n+1)$ -entry of the matrix

$$\prod_{i=1}^r ((1 + u_n)^{\chi(\gamma_i)} - 1)^{d_i} = \prod_{i=1}^r (\chi(\gamma_i) u_n)^{d_i} = \left(\prod_{i=1}^r \chi(\gamma_i)^{d_i} \right) u_n^n,$$

proving the lemma. □

The key lemma is then the following.

Lemma 4.5.2. *The image of $p_{-, n}: H^\vee \rightarrow (I^n/I^{n+1})^\vee$ generates $(I^n/I^{n+1})^\vee$.*

Proof. Using our identification (4.3), any non-zero $F \in I^n/I^{n+1}$ has a unique representative also denoted F in $\mathbb{F}_p[x_1, \dots, x_r]$ that is homogeneous of degree n and Lemma 4.5.1 implies that

$$p_{\chi, n}(F) = F(\chi(\gamma_1), \dots, \chi(\gamma_r)).$$

Writing $\Gamma: H^\vee \rightarrow \mathbb{F}_p^r$ for the isomorphism given by $\Gamma(\chi) = (\chi(\gamma_1), \dots, \chi(\gamma_r))$, this can be succinctly written as $p_{\chi,n}(F) = F(\Gamma(\chi))$.

For a finite set S and an $s \in S$, we denote by \mathbb{F}_p^S the vector space of functions $S \rightarrow \mathbb{F}_p$ and by $1_s \in \mathbb{F}_p^S$ the indicator function of s . The lemma may then be rephrased as the statement that the linearization $\tilde{p}_{-,n}$ of $p_{-,n}$, given by

$$\tilde{p}_{-,n}: \mathbb{F}_p^{H^\vee} \rightarrow (I^n/I^{n+1})^\vee, \quad 1_\chi \mapsto p_{\chi,n}$$

is surjective, or equivalently that the dual map

$$\tilde{p}_{-,n}^\vee: I^n/I^{n+1} \rightarrow (\mathbb{F}_p^{H^\vee})^\vee$$

is injective. For any non-zero $F \in I^n/I^{n+1}$, since $n < p$, the finite field Nullstellensatz provides the existence of $v \in \mathbb{F}_p^r$ for which $F(v) \neq 0$. Then

$$\tilde{p}_{-,n}^\vee(F)(1_{\Gamma^{-1}(v)}) = p_{\Gamma^{-1}(v),n}(F) = F(v) \neq 0,$$

so $\tilde{p}_{-,n}^\vee(F) \neq 0$. □

Remark 4.5.3. This lemma is the reason for our assumption that $n < p$ in this section rather than the assumption $n|R| < p|A|$ used in other sections. To see that this argument cannot work for torsion-free abelian groups H and arbitrary n , take $R = \mathbb{F}_p$ and $H \cong \mathbb{Z}_p^2$. Then we know that I^n/I^{n+1} has dimension $n + 1$ for any n , and the proof of Lemma 4.5.2 shows that $p_{-,n}: \text{Hom}(H, \mathbb{F}_p) \rightarrow (I^n/I^{n+1})^\vee$ is homogeneous of degree n in the sense that $p_{a\varphi,n} = a^n p_{\varphi,n}$ for $\varphi \in \text{Hom}(H, \mathbb{F}_p)$ and $a \in \mathbb{F}_p$, so the span of its image has dimension at most the cardinality of $\text{Hom}(H, \mathbb{F}_p)/\mathbb{F}_p^\times$, which is $p + 1$.

We now come to our result expressing values of the generalized Bockstein maps as sums of “cyclic” Massey products. If $\chi \in H^\vee$ and $f \in Z^1(G, T \otimes_R \Omega/I^n)$, then we say that a proper defining system relative to $\begin{bmatrix} \chi \\ n \end{bmatrix}$ is attached to f if it is attached to the image of f in $Z^1(G, T \otimes_R \Omega_\chi/I_\chi^n)$, where $\Omega_\chi = R[H/\ker \chi]$ and I_χ is its augmentation ideal.

Theorem 4.5.4. *There exist $N \geq 1$ and $\chi_1, \dots, \chi_N \in H^\vee$ such that $(p_{\chi_i,n})_{i=1}^N$ is an ordered \mathbb{F}_p -basis of $(I^n/I^{n+1})^\vee$. For any such $(\chi_i)_{i=1}^N$, let $(y_i)_{i=1}^N$ be the basis of I^n/I^{n+1} dual to $(p_{\chi_i,n})_{i=1}^N$. Then for any $f \in Z^1(G, T \otimes_R \Omega/I^n)$, we have*

$$\Psi^{(n)}(f) = \sum_{i=1}^N (\chi_i^{(n)}, \lambda)_{\rho_i} \cdot y_i,$$

where ρ_i is the proper defining system relative to $\begin{bmatrix} \chi_i \\ n \end{bmatrix}$ attached to f and λ is the image of f in $Z^1(G, T)$.

Proof. The first statement is clear from Lemma 4.5.2. For the second statement, let

$$\Psi^{(n)}(f) = \sum_{i=1}^N c_i \cdot y_i.$$

for some $c_i \in H^2(G, T)$. Since $p_{\chi_1, n}, \dots, p_{\chi_N, n}$ is the dual basis to y_1, \dots, y_N , we have $c_i = p_{\chi_i, n}(\Psi^{(n)}(f))$ for $1 \leq i \leq N$. But by Theorem 4.1.2, we have

$$p_{\chi_i, n}(\Psi^{(n)}(f)) = (\chi_i^{(n)}, \lambda)_{\rho_i}.$$

□

4.6 Heisenberg case

In this section, assume that $H = U_3(A)$ for a nonzero quotient A of \mathbb{Z}_p , and that R is a quotient of A such that either $R = \mathbb{Z}_p$ or $n|R| < p|A|$. We study the generalized Bockstein maps $\Psi^{(n)}$ in the cases $n = 2$ and $n = 3$.

Let

$$x = \left[\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] - 1, \quad y = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right] - 1, \quad z = \left[\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] - 1 \in \Omega. \quad (4.4)$$

Then I is the two-sided ideal generated by x and y , and $I/I^2 \cong Rx \oplus Ry$. Let $\chi, \psi: G \rightarrow A$ be the unique characters factoring through H and such that

$$\chi \left(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = 1, \quad \chi \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right) = 0, \quad \psi \left(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = 0, \quad \text{and} \quad \psi \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right) = 1.$$

Then $(\chi, \psi): G \rightarrow A \times A$ defines a homomorphism.

Lemma 4.6.1. *The R -module I^2/I^3 is freely generated by the image of the set*

$$S_2 = \{x^2, y^2, yx, z\},$$

and I^3/I^4 is R -freely generated by the image of

$$S_3 = \{x^3, xz, yx^2, y^2x, y^3, yz\}.$$

Proof. For any n , Lemma 2.3.1 and the condition that $n|R| < p|A|$ in the case that A is finite are enough to guarantee that the quotient Ω/I^{n+1} is isomorphic to the analogous quotient with A replaced by \mathbb{Z}_p , so we may suppose in this proof that $H = U_3(\mathbb{Z}_p)$.

Let Σ be the noncommutative $R[[z]]$ -power series ring Σ in variables x and y . It follows from the standard presentation of $U_3(\mathbb{Z}_p)$ as a finitely generated pro- p group that $\Omega = R[[U_3(\mathbb{Z}_p)]]$ is the quotient of Σ by the ideal generated by

$$w = (1 + y)(1 + x)z - (xy - yx). \quad (4.5)$$

The augmentation ideal I of Ω is (x, y) , so I^n is generated by the monomials in x and y of degree at least n . Using (4.5), we can reduce this to

$$I^n = (y^j x^i z^k \mid i + j + 2k \geq n).$$

It is therefore enough to check that the image of the set S_n is R -linearly independent in I^n/I^{n+1} for $n \in \{2, 3\}$.

Consider Σ as a graded R -algebra with x, y , and z in degrees 1, 1, and 2, respectively. Let J_n denote the ideal of elements of Σ of degree at least n . Suppose that $f \in \Sigma$ lies in the intersection of the R -span of the elements of S_n with $(w) + J_{n+1}$. When $n = 2$, one can easily see that $f = 0$. When $n = 3$, there are $a, b, c, d \in R$ such that

$$f + J_4 = (ax + by)w + w(cx + dy) + J_4.$$

By the hypothesis on f , the degree 3 terms above are in the R -span of S_3 , which forces $a = b = c = d = 0$, and hence $f = 0$. \square

Let us first consider the case $n = 2$. By Lemma 4.6.1, we see that I^2/I^3 is a free R -module on the set $S_2 = \{x^2, y^2, yx, z\}$. We consider the three partial defining systems

$$\phi_{x^2} = \begin{bmatrix} \chi \\ 2 \end{bmatrix}, \quad \phi_{y^2} = \begin{bmatrix} \psi \\ 2 \end{bmatrix}, \quad \phi_z: H \rightarrow U_3(R) \times U_1(R) = U_3(R),$$

with $a = 2$ and $b = 0$, where ϕ_z is the quotient map on coefficients, and the partial defining system

$$\phi_{yx} = (\chi, \psi): H \rightarrow U_2(R) \times U_2(R) = R \times R$$

for $a = b = 1$. By Theorem 3.3.4, the partial defining systems $\phi_{x^2}, \phi_{y^2}, \phi_z$, and ϕ_{yx} correspond to Massey products (χ, χ, \cdot) , (ψ, ψ, \cdot) , (χ, ψ, \cdot) , and (χ, \cdot, ψ) , respectively.

As for $n = 3$, the graded quotient I^3/I^4 is a free R -module on S_3 of Lemma 4.6.1. For each $s \in S_3$, we define a partial defining system ϕ_s (viewed as a pair of homomorphisms) as

follows:

$$\begin{aligned}
\phi_{x^3} &: H \xrightarrow{[\chi]_3, 0} U_4(R) \times U_1(R), \\
\phi_{xz} &: H \xrightarrow{\text{id}, \chi} U_3(R) \times U_2(R), \\
\phi_{yx^2} &: H \xrightarrow{\psi, [\chi]_2} U_2(R) \times U_3(R), \\
\phi_{y^2x} &: H \xrightarrow{[\psi]_2, \chi} U_3(R) \times U_2(R), \\
\phi_{y^3} &: H \xrightarrow{[\psi]_3, 0} U_4(R) \times U_1(R), \\
\phi_{yz} &: H \xrightarrow{\psi, \text{id}} U_2(R) \times U_3(R).
\end{aligned}$$

By Theorem 3.3.4, each partial defining system corresponds to a collection of Massey products as follows:

$$\begin{aligned}
\phi_{x^3} &\longleftrightarrow (\chi, \chi, \chi, \cdot), \\
\phi_{xz} &\longleftrightarrow (\chi, \psi, \cdot, \chi), \\
\phi_{yx^2} &\longleftrightarrow (\psi, \cdot, \chi, \chi), \\
\phi_{y^2x} &\longleftrightarrow (\psi, \psi, \cdot, \chi), \\
\phi_{y^3} &\longleftrightarrow (\psi, \psi, \psi, \cdot), \\
\phi_{yz} &\longleftrightarrow (\psi, \cdot, \chi, \psi).
\end{aligned}$$

For each $s \in S_n$ with $n \in \{2, 3\}$, the diagram (4.1) becomes

$$\begin{array}{ccccccc}
0 & \longrightarrow & T \otimes_R I^n/I^{n+1} & \longrightarrow & T \otimes_R \Omega/I^{n+1} & \longrightarrow & T \otimes_R \Omega/I^n & \longrightarrow & 0 \\
& & \downarrow p_s & & \downarrow p_s & & \downarrow p_s & & \\
0 & \longrightarrow & T & \longrightarrow & \mathfrak{U}_s(T) & \longrightarrow & \mathfrak{U}'_s(T) & \longrightarrow & 0,
\end{array}$$

where the maps p_s are induced by the map ϕ_s and we have used the shorthand $\mathfrak{U}_s(T)$ for $\mathfrak{U}_{\phi_s}(T)$ (and similarly for the quotients). Note that $p_s: T \otimes_R I^n/I^{n+1} \rightarrow T$ is just the R -tensor product of the likewise-defined $p_s: I^n/I^{n+1} \rightarrow R$ with the identity on T . The maps $p_s: I^n/I^{n+1} \rightarrow R$ for $s \in S_n$ form the dual basis to the R -basis S_n of I^n/I^{n+1} . This can be seen by an omitted direct computation, proceeding as in the following example.

Example 4.6.2. Suppose that $n = 2$, and take $s = z \in S_2$. Recall that $\phi_z: H \rightarrow U_3(R)$ is given by the canonical surjection $A \rightarrow R$ on coefficients. By definition of $p_z: \Omega/I^n \rightarrow \mathfrak{U}_{\phi_z}(R) = M_{3,1}(R)$ in Lemma 4.1.1, we have

$$p_z([h]) = \phi_z(h) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in M_{3,1}(R)$$

for all $h \in H$. Recalling that $x + 1$, $y + 1$, and $z + 1$ are the group elements of matrices as in (4.4), we compute

$$\begin{aligned} p_z(x^2) &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0, \\ p_z(y^2) &= \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0, \\ p_z(yx) &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0, \\ p_z(z) &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{4.6}$$

and note that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ gives the identity of $R \subset \mathfrak{U}_{\phi_z}(R)$.

By Theorem 4.1.2, we then have the following.

Theorem 4.6.3. *For $n \in \{2, 3\}$ and $f \in Z^1(G, T \otimes_R \Omega/I^n)$, the element $\Psi^{(n)}(f)$ of*

$$H^2(G, T) \otimes_R I^n/I^{n+1} \cong \bigoplus_{s \in S_n} H^2(G, T)_s$$

is the sum

$$(\chi, \chi, \lambda)_{\rho_{x^2}} x^2 + (\chi, \lambda, \psi)_{\rho_{yx}} yx + (\psi, \psi, \lambda)_{\rho_{y^2}} y^2 + (\chi, \psi, \lambda)_{\rho_z} z \tag{4.7}$$

for $n = 2$ and the sum

$$\begin{aligned} &(\chi, \chi, \chi, \lambda)_{\rho_{x^3}} x^3 + (\chi, \psi, \lambda, \chi)_{\rho_{xz}} xz + (\psi, \lambda, \chi, \chi)_{\rho_{yx^2}} yx^2 \\ &+ (\psi, \psi, \lambda, \chi)_{\rho_{y^2x}} y^2x + (\psi, \psi, \psi, \lambda)_{\rho_{y^3}} y^3 + (\psi, \lambda, \chi, \psi)_{\rho_{yz}} yz \end{aligned} \tag{4.8}$$

for $n = 3$, where each ρ_s for $s \in S_n$ is the proper defining system relative to ϕ_s attached to $p_s \circ f$ by Lemma 3.3.3, and λ is the image of f in $Z^1(G, T)$.

As before, Theorem 2.2.4 then provides the following isomorphisms.

Corollary 4.6.4. *Suppose that G is p -cohomologically finite of p -cohomological dimension 2. For $n \in \{2, 3\}$, let $P_n(H)$ denote the subgroup of $H^2(G, T) \otimes_R I^n/I^{n+1}$ consisting of all sums in (4.7) for $n = 2$ and in (4.8) for $n = 3$. We then have a canonical isomorphism of R -modules*

$$\frac{I^n H_{\text{Iw}}^2(N, T)}{I^{n+1} H_{\text{Iw}}^2(N, T)} \cong \frac{H^2(G, T) \otimes_R I^n/I^{n+1}}{P_n(H)}.$$

5 Massey vanishing for absolute Galois groups

In this section, we apply our results to the study of absolute Galois groups of fields.

5.1 The cyclic Massey vanishing property

Definition 5.1.1. Let G be a profinite group, and let p be a prime number. We say that G has the *p -cyclic Massey vanishing property* if for all homomorphisms $\chi, \lambda: G \rightarrow \mathbb{F}_p$ with $\chi \cup \lambda = 0$, there exists a proper defining system such that $(\chi^{(p-1)}, \lambda)$ vanishes.

As a simple corollary of [Sh1, Theorem 4.3], the absolute Galois group of field F containing a primitive p th root of unity has the p -cyclic Massey vanishing property. (For this, consider the case that Ω is the separable closure of K and $m = 1$ in the notation of said theorem.) The proof uses only the fact that if the norm residue symbol $(a, b)_{p, F}$ vanishes, then b is a norm from $F(a^{1/p})$. We shall give a streamlined proof of this and more, using the following abstract characterization of a standard property of absolute Galois groups.

Definition 5.1.2. Let $m \geq 1$, and set $R = \mathbb{Z}/m\mathbb{Z}$. We say that a profinite group G is of *m -absolute Galois type* if it has the property that, for any $\chi \in H^1(G, R)$, the sequence

$$H^1(G, R[H_\chi]) \rightarrow H^1(G, R) \xrightarrow{\chi \cup} H^2(G, R) \rightarrow H^2(G, R[H_\chi]) \quad (5.1)$$

is exact, where $H_\chi = G/\ker(\chi)$ is the coimage of χ .

Under Shapiro's lemma, the first and last maps in (5.1) are identified with corestriction and restriction maps, respectively [NSW, Proposition 1.6.5]. It is well-known that an absolute Galois group G_F is of m -absolute Galois type if F contains a primitive m th root of unity (see for instance [Se, Propositions XIV.2 and XIV.4]). This condition on G_F generalized to arbitrary cohomological degree is heavily used in the proof of the norm residue isomorphism theorem: see [HsWe, Theorem 3.6]. We focus on the comparison of p -absolute Galois type with p -cyclic Massey vanishing. In fact, our results would allow us to prove a more general but analogous result for profinite groups with the property that characters on G of order p^s lift to characters of order p^t for some large enough t relative to s , under conditions as in Section 4.3.

Proposition 5.1.3. *Let G be a profinite group. Then G has the p -cyclic Massey vanishing property if and only if the sequence (5.1) is exact at $H^1(G, \mathbb{F}_p)$.*

Proof. Let $\chi, \lambda: G \rightarrow \mathbb{F}_p$ with $\chi \cup \lambda = 0$, and set $\Omega = \mathbb{F}_p[H_\chi]$. The p th power of the augmentation ideal in Ω is zero, and the kernel of the generalized Bockstein map $\Psi^{(p-1)}$ is the image of $H^1(G, \Omega) \rightarrow H^1(G, \Omega/I^{p-1})$. Theorem 4.3.1 tells us that the Massey product $(\chi^{(p-1)}, \lambda)$ is defined and vanishes for some choice of proper defining system in $H^1(G, \Omega/I^{p-1})$ if and only if λ lifts to $H^1(G, \Omega)$. From this, we have the proposition. \square

Proposition 5.1.3 applies in particular to the absolute Galois group of any field F containing a primitive p th root of unity, i.e., G_F has the p -cyclic Massey vanishing property. We also have the following result, which may be of independent interest.

Proposition 5.1.4. *Let G be a profinite group. If (5.1) is exact at $H^2(G, \mathbb{F}_p)$ for a given $\chi \in H^1(G, \mathbb{F}_p)$, then it is exact at $H^1(G, \mathbb{F}_p)$, so G is of p -absolute Galois type.*

Proof. Let $\chi, \lambda: G \rightarrow \mathbb{F}_p$ with $\chi \cup \lambda = 0$, and suppose that (5.1) is exact at $H^2(G, \mathbb{F}_p)$. We have to show that there is a proper defining system ρ such that $(\chi^{(p-1)}, \lambda)_\rho$ vanishes. We may suppose that $\chi \neq 0$. Let $x = [h] - 1$ for $h \in H_\chi$ with $\chi(h) = 1$. By induction on n , we can assume that there is a proper defining system ρ_{x^n} for $(\chi^{(n)}, \lambda)$ with $n < p$ determined by some $f = \sum_{k=0}^{n-1} \lambda_k x^k \in Z^1(G, \Omega/I^n)$, with λ necessarily equal to λ_0 . Writing $(\chi^{(n)}, \lambda)_f$ for the corresponding Massey product $(\chi^{(n)}, \lambda)_{\rho_{x^n}}$, we have

$$(\chi^{(n)}, \lambda)_f = \chi \cup \lambda_{n-1} + \binom{\chi}{2} \cup \lambda_{n-2} + \cdots + \binom{\chi}{n} \cup \lambda.$$

Clearly the restriction of $(\chi^{(n)}, \lambda)_f$ to $\ker(\chi)$ vanishes, so, by the exactness of (5.1) at $H^2(G, \mathbb{F}_p)$, we have $(\chi^{(n)}, \lambda)_f = \chi \cup \psi$ for some $\psi \in H^1(G, \mathbb{F}_p)$. Then we see that $f' = f - \psi x^{n-1}$ is a proper defining system such that the Massey product $(\chi^{(n)}, \lambda)_{f'}$ vanishes. By Theorem 4.3.1, this implies that the class of f' is in the kernel of $\Psi^{(n)}$, so it lifts to the class of some $\tilde{f} \in Z^1(G, \Omega/I^{n+1})$, which gives rise to a proper defining system $\rho_{x^{n+1}}$ for $(\chi^{(n+1)}, \lambda)$. If $n + 1 = p$, then the class of \tilde{f} is the desired lift to $H^1(G, \Omega)$. \square

It is unclear that exactness of (5.1) at $H^1(G, \mathbb{F}_p)$ should imply exactness at $H^2(G, \mathbb{F}_p)$.

5.2 Triple Massey vanishing

In this subsection, let us suppose that p is an odd prime. The following theorem gives a new proof of the vanishing of triple Massey products for absolute Galois groups due to Efrat–Matzri [EfMa] and Mináč–Tân [MT3]. Both proofs used that the absolute Galois groups of a field containing a primitive p th root of unity is of p -absolute Galois type. We show that the potentially weaker condition of p -cyclic Massey vanishing suffices.

Theorem 5.2.1. *Let G be a profinite group with the p -cyclic Massey vanishing property for an odd prime p . Let $\chi, \psi, \lambda \in H^1(G, \mathbb{F}_p)$ be such that $\chi \cup \lambda = \lambda \cup \psi = 0$. Then there exists a defining system ρ for (χ, λ, ψ) such that the triple Massey product $(\chi, \lambda, \psi)_\rho$ vanishes.*

The case where χ and ψ are linearly dependent follows easily from the p -cyclic Massey vanishing property, so we can and do assume that $(\chi, \psi): G \rightarrow \mathbb{F}_p^2$ is surjective, and we let H

be the coimage. Let $\Omega = \mathbb{F}_p[H]$ and let $I \subset \Omega$ be the augmentation ideal. Let $h_\chi, h_\psi \in H$ be the dual basis to (χ, ψ) , and let $x = [h_\chi] - 1$ and $y = [h_\psi] - 1$ so that $I = x\Omega + y\Omega$.

We want to make maximal use of the fact that G has the cyclic Massey vanishing property. For this, we let C_1, C_2 , and C_3 be the coimages of $\alpha_1 = \chi$, $\alpha_2 = \psi$, and $\alpha_3 = \chi + \psi$, respectively. Let $\Omega_i = \mathbb{F}_p[C_i]$, and let $I_i \subset \Omega_i$ be its augmentation ideal. Let $\gamma_i \in C_i$ with $\alpha_i(\gamma_i) = 1$, and let $x_i = [\gamma_i] - 1 \in I_i$. Note that each α_i factors through H , so α_i induces a surjective ring homomorphism $\Omega \rightarrow \Omega_i$ that we also call α_i . Then note that

$$(\alpha_1(x), \alpha_1(y)) = (x_1, 0), \quad (\alpha_2(x), \alpha_2(y)) = (0, x_2), \quad (\alpha_3(x), \alpha_3(y)) = (x_3, x_3). \quad (5.2)$$

Now consider the ideal $J = I^3 + xy\Omega$, and let $J_i = \alpha_i(J)$. By (5.2), we have $J_1 = I_1^3$, $J_2 = I_2^3$, and $J_3 = I_3^2$. Hence we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & J/I^3 & \longrightarrow & I/I^3 & \longrightarrow & I/J \longrightarrow 0 \\ & & \downarrow \wr & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I_3^2/I_3^3 & \longrightarrow & \bigoplus_{i=1}^3 I_i/I_i^3 & \longrightarrow & \bigoplus_{i=1}^3 I_i/J_i \longrightarrow 0, \end{array} \quad (5.3)$$

where the vertical maps are induced by the maps α_i . Note that $J/I^3 = \mathbb{F}_p xy$, so the leftmost vertical arrow is an isomorphism, and $I/J = \mathbb{F}_p x \oplus \mathbb{F}_p x^2 \oplus \mathbb{F}_p y \oplus \mathbb{F}_p y^2$, and the map $I/J \rightarrow I_1/I_1^3 \oplus I_2/I_2^3$ is an isomorphism, so the rightmost vertical arrow is split injective.

Lemma 5.2.2. *There is a commutative diagram with exact rows*

$$\begin{array}{ccccccc} H^1(G, I/J) & \longrightarrow & H^2(G, J/I^3) & \xrightarrow{\iota} & H^2(G, I/I^3) & \longrightarrow & H^2(G, I/J) \\ \downarrow & & f \downarrow \wr & & g \downarrow & & \downarrow \\ H^1(G, \mathbb{F}_p) & \xrightarrow{\alpha_3 \cup} & H^2(G, \mathbb{F}_p) & \xrightarrow{h} & \bigoplus_{i=1}^3 H^2(G, I_i/I_i^3) & \longrightarrow & \bigoplus_{i=1}^3 H^2(G, I_i/J_i), \end{array} \quad (5.4)$$

where f is the isomorphism $\xi \cdot xy \mapsto \xi$ and g is the map induced by the center vertical arrow in (5.3).

Proof. The lower sequence in (5.3) is a direct sum of three exact sequences for $i \in \{1, 2, 3\}$, where for $i \in \{1, 2\}$ the sequence has zero as its first term. Taking cohomology of (5.3), we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc} H^1(G, I/J) & \longrightarrow & H^2(G, J/I^3) & \xrightarrow{\iota} & H^2(G, I/I^3) & \longrightarrow & H^2(G, I/J) \\ \downarrow & & \downarrow \wr & & g \downarrow & & \downarrow \\ \bigoplus_{i=1}^3 H^1(G, I_i/J_i) & \xrightarrow{\partial_3} & H^2(G, I_3^2/I_3^3) & \longrightarrow & \bigoplus_{i=1}^3 H^2(G, I_i/I_i^3) & \longrightarrow & \bigoplus_{i=1}^3 H^2(G, I_i/J_i), \end{array} \quad (5.5)$$

where ∂_3 is zero on the first two terms of the summand and the connecting map on the third. Note that the rightmost vertical arrow is injective by the split injectivity of the underlying map on coefficients. To complete the proof, we have to show that $\partial_3(\beta) = \alpha_3 \cup \beta$ for $\beta \in H^1(G, I_3/J_3)$. But the lower sequence in (5.3) for $i = 3$ is isomorphic to

$$0 \rightarrow I_3/I_3^2 \rightarrow \Omega_3/I_3^2 \rightarrow \mathbb{F}_p \rightarrow 0$$

via the isomorphism $I_3/J_3 \xrightarrow{\sim} \mathbb{F}_p$ taking the image of x_3 to 1, so this follows from Proposition 2.3.3. \square

Proof of Theorem 5.2.1. Consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J/I^3 & \longrightarrow & \Omega/I^3 & \longrightarrow & \Omega/J & \longrightarrow & 0 \\ & & \downarrow & & \parallel & & \downarrow & & \\ 0 & \longrightarrow & I/I^3 & \longrightarrow & \Omega/I^3 & \longrightarrow & \mathbb{F}_p & \longrightarrow & 0 \end{array}$$

and the associated diagram in cohomology

$$\begin{array}{ccccccccc} H^1(G, \Omega/I^3) & \longrightarrow & H^1(G, \Omega/J) & \xrightarrow{\partial'} & H^2(G, J/I^3) & \longrightarrow & H^2(G, \Omega/I^3) & & \\ \parallel & & \downarrow & & \downarrow \iota & & \parallel & & (5.6) \\ H^1(G, \Omega/I^3) & \longrightarrow & H^1(G, \mathbb{F}_p) & \xrightarrow{\partial} & H^2(G, I/I^3) & \longrightarrow & H^2(G, \Omega/I^3), & & \end{array}$$

where ι is as in (5.4).

Now let $\lambda \in H^1(G, \mathbb{F}_p)$ be as in the statement of the theorem and consider the element $\partial(\lambda) \in H^2(G, I/I^3)$. Then $g(\partial(\lambda)) \in \bigoplus_{i=1}^3 H^2(G, I_i/I_i^3)$ is the obstruction to lifting λ to $H^1(G, \Omega/I_i^3)$ for all $i \in \{1, 2, 3\}$, and this vanishes by the p -cyclic Massey vanishing property. Hence $g(\partial(\lambda)) = 0$.

By Lemma 5.2.2 and the injectivity of the rightmost vertical arrow in (5.4), this implies that $\partial(\lambda)$ is in the image of ι . By the commutativity of (5.6), there is then a lift $\tilde{\lambda} \in H^1(G, \Omega/J)$ of λ . Using Corollary 4.4.2, we see that $\tilde{\lambda}$ determines a proper defining system ρ_{xy} for (χ, λ, ψ) such that $\partial'(\tilde{\lambda}) = (\chi, \lambda, \psi)_{\rho_{xy}} \cdot xy$.

By Lemma 5.2.2, we have

$$hf(\partial'(\tilde{\lambda})) = g\iota(\partial'(\tilde{\lambda})) = g(\partial(\lambda)) = 0,$$

so $f(\partial'(\tilde{\lambda})) \in \ker(h) = \text{im}(\alpha_3 \cup)$. Hence we have

$$f(\partial'(\tilde{\lambda})) = (\chi, \lambda, \psi)_{\rho_{xy}} = \alpha_3 \cup \nu = \chi \cup \nu - \nu \cup \psi \quad (5.7)$$

in $H^2(G, \mathbb{F}_p)$, for some $\nu \in H^1(G, \mathbb{F}_p)$. In particular, we have that

$$(\chi, \lambda, \psi)_{\rho_{xy}} \in \text{im}(\chi \cup) + \text{im}(\cup \psi),$$

which implies that there is a defining system ρ such that $(\chi, \lambda, \psi)_\rho = 0$. \square

The reader may note that in Theorem 5.2.1, we used something weaker than p -cyclic Massey vanishing. Namely, the actual condition employed is that for any character $\chi: G \rightarrow \mathbb{F}_p$, the sequence

$$H^1(G, \mathbb{F}_p[H_\chi]/I_\chi^3) \rightarrow H^1(G, \mathbb{F}_p) \xrightarrow{\chi \cup} H^2(G, \mathbb{F}_p) \quad (5.8)$$

is exact, where $H_\chi = G/\ker(\chi)$ and I_χ is the augmentation ideal in $\mathbb{F}_p[H_\chi]$. This is equivalent to the statement that if $\chi \cup \lambda = 0$ for some $\lambda \in H^1(G, \mathbb{F}_p)$, then (χ, χ, λ) is zero for some proper defining system.

Remark 5.2.3. In [Ma, Corollary 3.5], Matzri proved that Massey triple vanishing follows from defined Massey products of the form (χ, λ, χ) containing zero. The proof exploits the exactness of (5.1) at $H^2(G, \mathbb{F}_p)$ to obtain this vanishing. From our perspective, the vanishing of these Massey products follows directly from the exactness of (5.8).

Remark 5.2.4. The proof of Theorem 5.2.1 does not show that G has the ‘‘bicyclic Massey vanishing property’’ that any $\lambda \in H^1(G, \mathbb{F}_p)$ that lifts to $H^1(G, \Omega/I^2)$ lifts further to $H^1(G, \Omega/I^3)$. Equivalently, this condition can be formulated as saying that if $\chi \cup \lambda = \lambda \cup \psi = 0$, then $\partial(\lambda) = 0$. One can show that this is equivalent to showing that there exists $\nu \in H^1(G, \mathbb{F}_p)$ satisfying (5.7) that lies in the subgroup

$$\ker(\chi \cup) + \ker(\psi \cup) + \ker((\chi + \psi) \cup).$$

A Two lemmas from homological algebra

We provide a proof of the following simple lemma from homological algebra for the reader’s convenience.

Lemma A.0.1. *Let \mathcal{Q} , \mathcal{R} , and \mathcal{S} be abelian categories such that \mathcal{Q} and \mathcal{R} have enough projectives, and let $F: \mathcal{R} \rightarrow \mathcal{S}$ and $F': \mathcal{Q} \rightarrow \mathcal{R}$ be right exact functors such that F' sends projective objects to F -acyclic objects. Let*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

be an exact sequence in \mathcal{Q} such that

$$0 \rightarrow F'(A) \rightarrow F'(B) \rightarrow F'(C) \rightarrow 0$$

is exact. For each $j \geq 0$, we have commutative diagrams

$$\begin{array}{ccc} L_{j+1}(F \circ F')(C) & \longrightarrow & L_j(F \circ F')(A) \\ \downarrow & & \downarrow \\ L_{j+1}F(F'(C)) & \longrightarrow & L_jF(F'(A)), \end{array}$$

in which the vertical arrows are edge maps in the Grothendieck spectral sequence attached to the composition $F \circ F'$ and the horizontal maps are connecting morphisms, where L_i denotes the i th left derived functor.

Proof. Let X denote any of A , B , and C . We may choose projection resolutions P_{\cdot}^X of X with each term of

$$0 \rightarrow F'(P_{\cdot}^A) \rightarrow F'(P_{\cdot}^B) \rightarrow F'(P_{\cdot}^C) \rightarrow 0$$

split exact. Then we may choose first quadrant Cartan-Eilenberg resolutions $Q_{\cdot, \cdot}^X$ of the $F'(P_{\cdot}^X)$ fitting in split exact sequences

$$0 \rightarrow Q_{j,k}^A \rightarrow Q_{j,k}^B \rightarrow Q_{j,k}^C \rightarrow 0$$

so that, in particular, we have exact sequences

$$0 \rightarrow H_k(Q_{j,\cdot}^A) \rightarrow H_k(Q_{j,\cdot}^B) \rightarrow H_k(Q_{j,\cdot}^C) \rightarrow 0,$$

and the complexes $H_k(Q_{\cdot, \cdot}^X) \rightarrow H_k(F'(P_{\cdot}^X))$ are projective resolutions. Note that

$$H_j(F(H_k(Q_{\cdot, \cdot}^X))) = L_jF(L_kF'(X)),$$

and we have canonical isomorphisms

$$H_j(F(\text{Tot } Q_{\cdot, \cdot}^X)) \xrightarrow{\sim} H_j(F \circ F'(P_{\cdot}^X)) = L_j(F \circ F')(X),$$

the first isomorphism as the terms of $F'(P_{\cdot}^X)$ are F -acyclic. The diagram in question is then simply

$$\begin{array}{ccc} H_{j+1}(F(\text{Tot } Q_{\cdot, \cdot}^C)) & \longrightarrow & H_j(F(\text{Tot } Q_{\cdot, \cdot}^A)) \\ \downarrow & & \downarrow \\ H_{j+1}(F(H_0(Q_{\cdot, \cdot}^C))) & \longrightarrow & H_j(F(H_0(Q_{\cdot, \cdot}^A))), \end{array}$$

the horizontal arrows being the connecting homomorphisms and the vertical arising from the augmentation maps on the total complexes. \square

The following lemma is rather elementary but also useful to us.

Lemma A.0.2. *Let \mathcal{R} and \mathcal{S} be abelian categories such that \mathcal{R} has enough projectives. Let $F: \mathcal{R} \rightarrow \mathcal{S}$ be a left exact functor. Suppose that $G: \mathcal{R} \rightarrow \mathcal{S}$ is a functor such that the pair (F, G) extends to a functor from short exact sequences $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathcal{R} to exact sequences*

$$G(A) \rightarrow G(B) \rightarrow G(C) \xrightarrow{\delta} F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0.$$

Then there is a natural transformation $G \rightsquigarrow L_1F$ such that the resulting diagrams

$$\begin{array}{ccc} G(C) & & \\ \downarrow & \searrow \delta & \\ L_1F(C) & \xrightarrow{\partial} & F(A), \end{array}$$

are commutative for the usual connecting homomorphisms ∂ and such that $G(A) \rightarrow L_1F(A)$ is an epimorphism for all objects A of \mathcal{R} .

Proof. Put any object A of \mathcal{R} in an exact sequence

$$0 \rightarrow K \rightarrow P \rightarrow A \rightarrow 0$$

in \mathcal{R} , where P is a projective object. We then have a commutative diagram

$$\begin{array}{ccccccc} G(P) & \longrightarrow & G(A) & \longrightarrow & F(K) & \longrightarrow & F(P) \\ & & \downarrow & & \parallel & & \parallel \\ 0 & \longrightarrow & L_1F(A) & \longrightarrow & F(K) & \longrightarrow & F(P), \end{array}$$

with exact rows, where the vertical morphism is unique making the diagram commute. That this gives a natural transformation is standard, and the fact that the morphisms are epimorphisms follows from the five lemma. \square

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