Problem 1 : Classification of Quadratic Form

Firstly use completing the square to classify the following quadratic forms;
then use $4AC - B^2$ method to do it again;
finally determine the zero sets for each of them:

- The first 5 is in good form so no need to complete the square
- $4AC - B^2 < 0$ indefinite
- $4AC - B^2 = 0$ semi-definite
- $4AC - B^2 > 0$ $\begin{cases} A > 0 & \text{positive definite} \\ A < 0 & \text{negative definite} \end{cases}$

zero sets on last page:

1. $f(x,y) = x^2 + 2y^2$ positive definite
2. $f(x,y) = x^2 - y^2$ indefinite
3. $f(x,y) = -x^2 - y^2$ negative definite
4. $f(x,y) = xy$ indefinite
5. $f(x,y) = x^2$ semi-definite
6. $f(x,y) = x^2 - 4xy + 3y^2 = x^2 + 2\cdot x\cdot(-2y) + (-2y)^2 - (-2y)^2 + 3y^2 = [x + (-2y)]^2 - y^2$ indefinite
7. $f(x,y) = 9x^2 - 36xy + 81y^2 = 9[x^2 - 4xy + 9y^2]$
   $= 9[x^2 + 2\cdot x\cdot(-2y) + (-2y)^2 - (-2y)^2 + 9y^2]$
   $= 9[(x-2y)^2 + 5y^2] = 9(x-2y)^2 + 45y^2$
   positive definite
8. $f(x,y) = xy + y^2$
9. $f(x,y) = x^2 + 2xy$
10. $f(x,y) = \frac{1}{2}x^2 - xy + y^2$

Problem 2: Domain

Find the largest domain where the functions can be defined:

1. $f(x,y) = \sqrt{9 - x^2} + \sqrt{y^2 - 4}$
2. $f(x,y) = \frac{1}{\sqrt{16 - x^2 - 4y^2}}$

1. $\begin{cases} 9 - x^2 \geq 0 \\ y^2 - 4 \geq 0 \end{cases} \Rightarrow \begin{cases} |x| \leq 3 \\ |y| \geq 2 \end{cases}$

2. $16 - x^2 - 4y^2 > 0$

8. $xy + y^2$
$= y^2 + 2\cdot y\cdot \frac{x}{2} + (\frac{x}{2})^2 - (\frac{x}{2})^2$
$= (y + \frac{x}{2})^2 - (\frac{x}{2})^2$ indefinite

9. $x^2 + 2xy$
$= x^2 + 2\cdot x\cdot y + y^2 - y^2$
$= (x+y)^2 - y^2$
indefinite

10. $\frac{1}{2}x^2 - xy + y^2$
$= \frac{1}{2}\cdot[x^2 - 2xy + 2y^2]$
$= \frac{1}{2}[x^2 + 2\cdot x\cdot(-y) + (-y)^2 - (-y)^2 + 2y^2]$
$= \frac{1}{2}[(x-y)^2 + y^2]$ positive
$= \frac{1}{2}(x-y)^2 + \frac{1}{2}y^2$ definite

Problem 3: Limit for Two Variable Function

Determine if the following functions have a limit as (x, y) approaches $(0, 0)$:

1. $f(x,y) = \frac{xy}{x^2+y^2}$ No
2. $f(x,y) = \frac{x^2}{x^2+y^2}$ No
3. $f(x,y) = \frac{x^2}{\sqrt{x^2+y^2}}$ Yes

1. Restrict the function to $y = kx$

$$f(x,y) = \frac{x \cdot kx}{x^2 + k^2x^2} = \frac{k}{1+k^2}$$

So $\lim_{x \to 0} f(x, kx) = \frac{k}{1+k^2}$

Choosing different k gives different limit. So there is no limit for $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

2. Restrict the f to $y = kx$

So $f(x,y) = \frac{1}{1+k^2}$

Choosing different k gives different limit. So no limit.

3. $f(x,y) = \frac{r^2\cos^2\theta}{r} = r\cos^2\theta \le r$

So $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} r\cos^2\theta = 0$

Problem 4: Partial Derivative

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for each of the following functions:

Before you start, please remind yourself of Product Rule, Quotient Rule, Chain Rule and derivatives for x^n, e^x, $\ln x$, $\sin x$, $\cos x$.

1. $f(x,y) = x^2y^3 - x^3y^2$
2. $f(x,y) = \cos(x^y) + y^3$
3. $f(x,y) = \frac{xy}{x^2+y}$
4. $f(x,y) = e^{x^2+y^2}$
5. $f(x,y) = xy \ln xy$
6. $f(x,y,z) = \sqrt{x^2+y^2+z^2}$
7. $f(x,y) = x \tan y$

1. $\frac{\partial f}{\partial x} = 2x \cdot y^3 - 3x^2 \cdot y^2$

$\frac{\partial f}{\partial y} = x^2 \cdot 3y^2 - x^3 \cdot 2y$

2. $\frac{\partial f}{\partial x} = -\sin(x^y) \cdot y x^{y-1}$

$\frac{\partial f}{\partial y} = -\sin(x^y) \cdot x^y \ln x$

3. $\frac{\partial f}{\partial x} = \frac{y \cdot (x^2+y) - xy \cdot 2x}{(x^2+y)^2}$

$= \frac{y^2 - x^2 \cdot y}{(x^2+y)^2}$

$\frac{\partial f}{\partial y} = \frac{x \cdot (x^2+y) - xy}{(x^2+y)^2}$

$= \frac{x^3}{(x^2+y)^2}$

4. $\frac{\partial f}{\partial x} = e^{x^2+y^2} \cdot 2x$

$\frac{\partial f}{\partial y} = e^{x^2+y^2} \cdot 2y$

5. $\frac{\partial f}{\partial x} = y \cdot \ln xy + xy \cdot \frac{1}{xy} \cdot y$

$= y \ln xy + y$

$\frac{\partial f}{\partial y} = x \ln xy + x$

6. $\frac{\partial f}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}}$

$= \frac{x}{\sqrt{x^2+y^2+z^2}}$

$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$

$\frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$

7. $\frac{\partial f}{\partial x} = \tan y$

$\frac{\partial f}{\partial y} = x \cdot \sec^2 y$

Problem 1:

zero set :

1. $x^2+2y^2=0 \Rightarrow x=0 \quad y=0$

2. $x^2-y^2=0=(x+y)(x-y) \Rightarrow x+y=0$ or $x-y=0$

3. $-x^2-y^2=0 \Rightarrow x=0 \quad y=0$

4. $xy=0 \Rightarrow x=0$ or $y=0$

5. $x^2=0 \Rightarrow x=0$

6. $[x+(-2y)]^2-y^2=(x-3y)(x-y) \Rightarrow x-3y=0$ or $x-y=0$

7. $9(x-2y)^2+45y^2=0 \Rightarrow x-2y=0$ and $y=0 \Rightarrow x=0 \; y=0$

8. $(y+\frac{x}{2})^2-(\frac{x}{2})^2=y\cdot(x+y) \Rightarrow y=0$ or $x+y=0$

9. $x^2+2xy=x\cdot(x+2y)=0 \Rightarrow x=0$ or $x+2y=0$

10. $\frac{1}{2}(x-y)^2+\frac{1}{2}y^2=0 \Rightarrow x-y=0$ and $y=0 \Rightarrow x=0 \; y=0$

Summary:

Type	General Form	$4AC-B^2$	Zero Set	Function Value
Indefinite	$+(\quad)^2-(\quad)^2$	$-$	2 lines	Positive and Negative
Semi-definite	$(\quad)^2$ or $-(\quad)^2$	0	1 line	Positive or Negative
Positive-definite	$+(\quad)^2+(\quad)^2$	$+$ $(A>0)$	$(0,0)$	Positive
Negative-definite	$-(\quad)^2-(\quad)^2$	$+$ $(A<0)$	$(0,0)$	Negative

☆: To tell between Positive-definite and Negative-definite, you look at sign of A