Problem 1 : Tangent Plane/Linear Approximation
For following surfaces:
1) compute the partial derivatives;
2) write up the differential df (i.e. linear approximations);
3) find out the tangent plane at the given point;
4) what is the normal vector of the plane you find out.

1. $z = xy^2$; $x = 2, y = 1$
2. $z = \frac{xy}{x+y}$; $x = 3, y = 1$
3. $x^2 + y^2 + z^2 = 3$; $x = 1, y = 1, z = 1$; $x = 1, y = 1, z = -1$

1. 1) $\frac{\partial z}{\partial x} = y^2 \qquad \frac{\partial z}{\partial y} = 2xy$

2) $df = y^2 \cdot dx + 2xy\, dy$

3) $df = dx + 4dy$ at $(2,1,2)$

$(z-2) = (x-2) + 4\cdot(y-1)$

$\Leftrightarrow \quad x + 4y - z = 4$

4) $\vec{n} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$

2. 1) $\frac{\partial z}{\partial x} = \frac{y^2}{(x+y)^2} \qquad \frac{\partial z}{\partial y} = \frac{x^2}{(x+y)^2}$

2) $df = \frac{y^2}{(x+y)^2} dx + \frac{x^2}{(x+y)^2} dy$

3) at $(3, 1, \frac{3}{4})$

$df = \frac{1}{16} dx + \frac{9}{16} dy$

$z - \frac{3}{4} = \frac{1}{16}\cdot(x-3) + \frac{9}{16}(y-1)$

4) $\vec{n} = \begin{pmatrix} \frac{1}{16} \\ \frac{9}{16} \\ -1 \end{pmatrix}$

3. 1) $z = \sqrt{3-x^2-y^2}$

for $(1,1,1)$

$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{3-x^2-y^2}} \qquad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{3-x^2-y^2}}$

2) $df = \frac{-x}{\sqrt{3-x^2-y^2}} dx + \frac{-y}{\sqrt{3-x^2-y^2}} dy$

3) $df = -dx - dy$ at $(1,1,1)$

$z - 1 = -(x-1) - (y-1) \Leftrightarrow x + y + z = 1$

4) $\vec{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

For points $(1, 1, -1)$

1) $z = -\sqrt{3-x^2-y^2}$

$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{3-x^2-y^2}}$

$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{3-x^2-y^2}}$

2) $df = \frac{x}{\sqrt{3-x^2-y^2}} dx + \frac{y}{\sqrt{3-x^2-y^2}} dy$

3) $df = dx + dy$

$z + 1 = x - 1 + y - 1$

$\Leftrightarrow \quad x + y - z = 3$

4) $\vec{n} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Problem 2: Chain Rule
Firstly use chain rule to compute $\frac{df}{dt}$, and then evaluate $\frac{df}{dt}$ at given point:

1. $f(x, y) = x^2y^3 + x^3y^2$; $x(t) = t^2 + t$, $y(t) = e^t$; $t = 0$;
2. $f(x, y) = x^2 + y^2$; $x(t) = \cos t$, $y(t) = 2\sin t$; $t = \frac{\pi}{2}$;
3. $f(x, y, z) = xyz$; $x(t) = \ln t$, $y(t) = e^t$, $z(t) = \frac{1}{t}$; $t = 2$;

1. $\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$

$= (2xy^3 + 3x^2y^2)\cdot(2t+1) + (3y^2x^2 + 2yx^3)\cdot e^t$

$\left.\frac{df}{dt}\right|_{t=0} = 0$

2. $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = 2x\cdot(-\sin t) + 2y\cdot(2\cos t)$

$\left.\frac{df}{dt}\right|_{t=\frac{\pi}{2}} = 2\cdot 0\cdot -1 + 2\cdot 2\cdot 0 = 0$

(because $x(\frac{\pi}{2}) = 0 \quad y(\frac{\pi}{2}) = 2$)

3. $\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt} = yz\cdot\frac{1}{t} + xz\cdot e^t + xy\cdot\frac{-1}{t^2}$

$\left.\frac{df}{dt}\right|_{t=2} = e^2\cdot\frac{1}{2}\cdot\frac{1}{2} + \ln 2\cdot\frac{1}{2}\cdot e^2 + \ln 2\cdot e^2\cdot\frac{-1}{4} = e^2\cdot\frac{1}{2}\cdot\ln 2$

Problem 3: Gradient

Compute the gradient of the following functions
then determine that at the given point, in which direction the function **increases** the fastest, and in which direction the function **decreases** the fastest, and in which direction the function **remains** the same ?

1. $f(x,y) = x^2 + 3xy^2$; $x = 1, y = 1$
2. $f(x,y) = 100 - x^2 - 3y^3$; $x = 2, y = 1$

1. $\nabla f = \begin{pmatrix} 2x + 3y^2 \\ 6xy \end{pmatrix}$ $\quad \nabla f|_{(1,1)} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

so in direction $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$ f increases the fastest, in direction $\begin{pmatrix} -5 \\ -6 \end{pmatrix}$ f decreases the fastest, in direction $\begin{pmatrix} 6 \\ -5 \end{pmatrix}$ or $\begin{pmatrix} -6 \\ 5 \end{pmatrix}$ f remains the same.

2. $\nabla f = \begin{pmatrix} -2x \\ -9 \cdot y^2 \end{pmatrix}$ $\quad \nabla f|_{(2,1)} = \begin{pmatrix} -4 \\ -9 \end{pmatrix}$

So in direction $\begin{pmatrix} -4 \\ -9 \end{pmatrix}$, ... increases ...
in direction $\begin{pmatrix} 4 \\ 9 \end{pmatrix}$, ... decreases ...... in direction $\begin{pmatrix} -9 \\ 4 \end{pmatrix}$ or $\begin{pmatrix} 9 \\ -4 \end{pmatrix}$, ... remains ...

Problem 4: Tangent Plane Revisited

We learned that: given $f(x,y,z)$, the level set is a surface, and the gradient is perpendicular to the tangent plane of the level set. Please use this idea to compute the tangent plane of the following surface:

1. $x^2 + 3y^2 + z^3 = 5$; $x = 1, y = 1, z = 1$
2. $x^2 + y^2 + z^2 = 1$; $x = 1, y = 0, z = 0$

1. Consider the surface as level set at 0 of
$f(x,y,z) = x^2 + 3y^2 + z^3 - 5$

then. $\nabla f = \begin{pmatrix} 2x \\ 6y \\ 3z^2 \end{pmatrix}$ $\quad \nabla f|_{(1,1,1)} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$ is the normal vector of tangent plane.

so that. the plane is $\begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} = 2(x-1) + 6 \cdot (y-1) + 3(z-1) = 0$

$\Leftrightarrow \quad 2x + 6y + 3z = 11$

2. Consider the surface as level set at 0 of
$f(x,y,z) = x^2 + y^2 + z^2 - 1$

$\nabla f = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$ $\quad \nabla f|_{(1,0,0)} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ is the normal vector of tangent plane.

so the plane is $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y-0 \\ z-0 \end{pmatrix} = 2 \cdot (x-1) + 0 + 0 = 0$

$\Leftrightarrow \quad x = 1$