Problem 1 : Implicit Function Derivative
For the following functions:
1.using implicit function method to compute the partial derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$
2.at which points the partial derivatives $\frac{\partial z}{\partial x}$ are not defined.

1. $x^2+y^2+z^2=1$
2. $x^2y+y^2z+z^2x=0$
3. $e^z+x^2z+y^2z=2$

$\frac{\partial z}{\partial x}$ is defined if $F_z = z \neq 0$

so the pts $\frac{\partial z}{\partial x}$ not defined is $x^2+y^2=1$

1. $F(x,y,z) = x^2+y^2+z^2-1=0$

$F_x = 2x \quad F_y = 2y \quad F_z = 2z$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{x}{z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{y}{z}$$

2. $F(x,y,z) = x^2y+y^2z+z^2x=0$

$F_x = 2xy+z^2,\ F_y = x^2+2yz;\ F_z = y^2+2zx$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2xy+z^2}{y^2+2zx} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x^2+2yz}{y^2+2zx}$$

$F_z = y^2+2zx=0$. then $\frac{\partial z}{\partial x}$ is not defined

3. $F(x,y,z) = e^z+x^2z+y^2z-2=0$

$F_x = 2xz \quad F_y = 2yz \quad F_z = e^z+y^2$

$$\frac{\partial z}{\partial x} = -\frac{2xz}{e^z+y^2} \qquad \frac{\partial z}{\partial y} = -\frac{2yz}{e^z+y^2}$$

$e^z+y^2=0$ then $\frac{\partial z}{\partial x}$ is not defined

Problem 2: Chain Rule Use chain rule to compute $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ (and $\frac{\partial f}{\partial w}$):

1. $f(x,y)=\sin(x^2+y^2)$; $x(u,v)=u^2-v^2$; $y(u,v)=2uv$;
2. $f(x,y)=x^2y$; $x(u,v)=\sin uv$, $y(u,v)=e^{uv}$;
3. $f(x,y,z) = \sqrt{x^2+y^2+z^2}$; $x(u,v) = u\sin v\cos w$, $y(u,v) = u\sin v\sin w$, $z(u,v) = u\cos v$;

1. $$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = 2x\cdot\cos(x^2+y^2)\cdot 2u + 2y\cdot\cos(x^2+y^2)\cdot(2v)$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} = 2x\cdot\cos(x^2+y^2)\cdot(-2v) + 2y\cdot\cos(x^2+y^2)\cdot(2u)$$

2. $$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = 2xy\cdot\cos(uv)\cdot v + x^2\cdot e^{uv}\cdot v$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} = 2xy\cdot\cos(uv)\cdot u + x^2\cdot e^{uv}\cdot u$$

3. $$\frac{\partial f}{\partial u} = f_x\cdot X_u + f_y\cdot Y_u + f_z\cdot Z_u = \frac{x}{\sqrt{x^2+y^2+z^2}}\cdot\sin v\cos w + \frac{y}{\sqrt{x^2+y^2+z^2}}\cdot\sin v\sin w + \frac{z}{\sqrt{x^2+y^2+z^2}}\cdot\cos v$$

$$\frac{\partial f}{\partial v} = \frac{1}{\sqrt{x^2+y^2+z^2}}\cdot[x\cdot u\cos w\cdot\cos v + y\cdot u\sin w\cos v + z\cdot u\cdot(-\sin v)]$$

$$\frac{\partial f}{\partial w} = \frac{1}{\sqrt{x^2+y^2+z^2}}\cdot[x\cdot u\sin v\cdot(-\sin w) + y\cdot u\sin v\cdot\cos w]$$

Problem 3 : Higher Partial Derivatives
For following functions:
compute higher order partial derivatives;

1. $f(x, y) = x^3y^2 + y^5$;
2. $f(x, y) = x \sin y$
3. $x^2 + 4y^2 + 16z^2 - 64 = 0$ where $z = z(x, y)$ is an implicit function of x and y.

1. $f_x = 3x^2y^2 \quad f_y = 2yx^3 + 5y^4$
$f_{xx} = 6xy^2 \quad f_{xy} = f_{yx} = 6x^2y$
$f_{yy} = 2x^3 + 20y^3$

2. $f_x = \sin y \quad f_y = x \cdot \cos y$
$f_{xx} = 0 \quad f_{xy} = f_{yx} = \cos y$
$f_{yy} = -x \cdot \sin y$

3. $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{32z} = -\frac{x}{16z}$
$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{8y}{32z} = -\frac{y}{4z}$
$z_{xx} = (-\frac{1}{16}) \cdot (\frac{x}{z})_x = -\frac{1}{16} \cdot \frac{1 \cdot z - x \cdot z_x}{z^2} = -\frac{z - x(\frac{-x}{16z})}{16z^2} = -\frac{16z^2 + x^2}{16^2z^3}$
$z_{xy} = (-\frac{x}{16}) \cdot (\frac{1}{z})_y = -\frac{x}{16} \cdot -\frac{1}{z^2} \cdot z_y$
$= \frac{x}{16z^2} \cdot (-\frac{y}{4z}) = \frac{-xy}{64z^3}$
$z_{yy} = (-\frac{1}{4}) \cdot (\frac{y}{z})_y$
$= (-\frac{1}{4}) \cdot \frac{1 \cdot z - y \cdot z_y}{z^2}$
$= -\frac{1}{4} \cdot \frac{z - y \cdot (-\frac{y}{4z})}{z^2}$
$= -\frac{4z^2 + y^2}{4^2z^3}$

Problem 4: Chain Rule with Higher Partial Derivatives
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15. a) $g_u = f_x \cdot x_u + f_y \cdot y_u$
$= f_x \cdot 1 + f_y \cdot 1 = f_x + f_y$
$g_{uu} = (g_u)_x \cdot x_u + (g_u)_y \cdot y_u$
$= (f_{xx} + f_{yx}) + (f_{xy} + f_{yy})$

b) $g_v = f_x \cdot x_v + f_y \cdot y_v = f_x \cdot 1 + f_y \cdot (-1) = f_x - f_y$
$g_{vv} = (g_v)_x \cdot x_v + (g_v)_y \cdot y_v = (f_{xx} - f_{yx}) - (f_{xy} - f_{yy})$
$= f_{xx} - f_{yx} - f_{xy} + f_{yy}$

c) $g_{uv} = (g_u)_x \cdot x_v + (g_u)_y \cdot y_v$
$= (f_{xx} + f_{yx}) - (f_{xy} + f_{yy}) = f_{xx} - f_{yy}$

d) $g_{uu} - g_{vv} = 4f_{xy}$

e) $g_{uu} + g_{vv} = 2f_{xx} + 2f_{yy}$