Problem 1 : Critical Points

For following functions:

1. Compute first order and second order derivatives;
2. Compute the critical points;
3. Classify the quadratic form $f_{xx}X^2 + 2f_{xy}XY + f_{yy}Y^2$ at critical points.

1. $f(x,y) = x^2 + 4y^2 - 2x + 8y - 1$;
2. $f(x,y) = (x-y)(xy-4)$;
3. $f(x,y) = y^2 + \cos x$;

1. $f_x = 2x-2$ $f_y = 8y+8$

$f_{xx} = 2$ $f_{xy} = 0$ $f_{yy} = 8$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{matrix} x=1 \\ y=-1 \end{matrix}$ c.p: $(1,-1)$

$Q(X,Y) = 2X^2 + 8Y^2$ positive definite

2. $f_x = 2xy - y^2 - 4$

$f_y = -2xy + x^2 + 4$

$f_{xx} = 2y$ $f_{xy} = 2x - 2y$

$f_{yy} = -2x$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow f_x + f_y = 0 \Rightarrow$

$x^2 = y^2$

① $x = y$ $f_x = x^2 - 4 = 0$

$\Rightarrow x = \pm 2$

c.p. $(2,2)$ $(-2,-2)$

② $x = -y$ no solution.

$Q(X,Y) = 4X^2 - 4Y^2$

at $(2,2)$ indefinite

$Q(X,Y) = 4Y^2 - 4X^2$

at $(-2,-2)$. indefinite

3. $f_x = -\sin x$

$f_y = 2y$

$f_{xx} = -\cos x$ $f_{yy} = 2$

$f_{xy} = 0$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{matrix} x = n\pi \\ y = 0 \end{matrix}$

c.p: $(n\pi, 0)$

n is arbitrary integer.

① if n is even,

$f_{xx} = -1$

$Q(X,Y) = -X^2 + 2Y^2$

indefinite

② if n is odd

$f_{xx} = 1$

$Q(X,Y) = X^2 + 2Y^2$

positive definite.

Problem 2 : Linear Regression

Given the following sample points, try to find the optimal linear model using linear regression: $(-1,0)$, $(0,2)$, $(1,4)$, $(2,5)$

$N = 4$

Error defined to be:

$$E(a,b) = \frac{1}{2}\sum_{k=1}^{4}(y_k - ax_k - b)^2$$

$$\frac{\partial E}{\partial a} = \frac{1}{2}\sum_{k=1}^{4} 2\cdot(y_k - ax_k - b)\cdot(-x_k)$$

$$= \left(\sum_{k=1}^{4} x_k^2\right)\cdot a + \left(\sum_{k=1}^{4} x_k\right) b - \left(\sum_{k=1}^{4} x_k\cdot y_k\right) = 6a + 2b - 14$$

$$\frac{\partial E}{\partial b} = \frac{1}{2}\cdot\sum_{k=1}^{4} 2(y_k - ax_k - b)\cdot(-1)$$

$$= \left(\sum_{k=1}^{4} x_k\right) a + 4b - \left(\sum_{k=1}^{4} y_k\right) = 2a + 4b - 11$$

Solve for c.p.

$$\begin{cases} \frac{\partial E}{\partial a} = 0 \\ \frac{\partial E}{\partial b} = 0 \end{cases} \Rightarrow \begin{cases} 6a + 2b = 14 \\ 2a + 4b = 11 \end{cases} \Rightarrow \begin{matrix} a = 1.7 \\ b = 1.9 \end{matrix}$$

So $y = 1.7x + 1.9$ is the best fit.