MATH 234 WORKSHEET 8 25 OCT 2016, TA: JIUYA WANG

Problem 1 : Maxima and Minima

For following functions:

1. compute the critical points;
2. Apply the second derivative test to find out local behavior;
3. If the second derivative test fails, use other method (drawing zero set or level set) to look for local behavior.

1. $f(x,y) = y^4 - 4*y^2 - 18x^2 + x^4$;
2. $f(x,y) = x(x-y)(x-1)$;
3. $f(x,y) = (1-x^2-y^2)^2$;
4. $f(x,y) = y(x+y)(x-y)$;

1. $f_x = -36x + 4x^3$ $f_{xx} = -36 + 12x^2$

$f_y = 4y^3 - 8y$ $f_{yy} = 12y^2 - 8$

$f_{xy} = 0$

so $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{matrix} x = 0, \pm 3 \\ y = 0, \pm 2 \end{matrix}$ cp: $(0,0)$ $(0,2)$ $(0,-2)$ $(3,0)$ $(3,2)$ $(3,-2)$ $(-3,0)$ $(-3,2)$ $(-3,-2)$

if $x=0$ $y=0$ $Q(X,Y) = -36X^2 - 8Y^2$ negative def local max.

if $x=0$ $y=\pm 2$ $Q(X,Y) = -36X^2 + 40Y^2$ indef. saddle pt

if $x=\pm 3$ $y=0$ $Q(X,Y) = 72X^2 - 8Y^2$ indef saddle pt.

if $x=\pm 3$ $y=\pm 2$ $Q(X,Y) = 72X^2 + 40Y^2$ positive def local min.

2. $f_x = (x-y)(x-1) + x\cdot(x-1) + x\cdot(x-y)$

$f_y = x\cdot(x-1)\cdot(-1)$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow$ $x = 0 = y$ or $x = y = 1$

$f_{xx} = 2(x-y) + 2(x-1) + 2x$

$f_{yy} = 0$

$f_{xy} = (-1)\cdot(2x-1)$

if $x=y=0$ $Q(X,Y) = -2X^2 + 2XY$ indefinite. saddle pt

if $x=y=1$ $Q(X,Y) = 2X^2 + 2XY$ indefinite. saddle pt

Problem 2 : Lagrange Multiplier

Find all the optimal points for the following questions by Lagrange Multiplier.

1. Page 104 Problem 1
2. Page 104 Problem 5 a
3. Page 104 Problem 8

3. $f_x = 2\cdot(1-x^2-y^2)\cdot(-2x)$

$f_y = 2\cdot(1-x^2-y^2)\cdot(-2y)$

$f_{xx} = 2\cdot(1-x^2-y^2)\cdot(-2) + 2\cdot(-2x)\cdot(-2x)$

$f_{xy} = 2\cdot(-2y)\cdot(-2x)$

$f_{yy} = 2\cdot(1-x^2-y^2)\cdot(-2) + 2\cdot(-2y)\cdot(-2y)$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow$ $1-x^2-y^2 = 0$ or $(0,0)$

if $x=0$ $y=0$ $Q(X,Y) = -4X^2 - 4Y^2$ negative definite local max

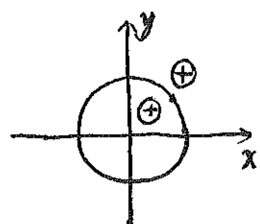
For (α,β) on unit sphere. $1-\alpha^2-\beta^2 = 0$

$Q(X,Y) = 8\alpha^2X^2 + 16\cdot\alpha\beta XY + 8\cdot\beta^2Y^2$

$= 8\cdot(\alpha X + \beta Y)^2$ semi-definite

inclusive :

Draw level set at $f(\alpha,\beta) = 0$.



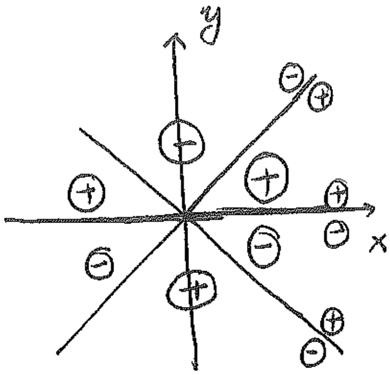
so local minimum.

4. $f_x = 2yx$ $f_y = x^2 - y^2 + 2xy$

$f_{xx} = 2y$ $f_{xy} = 2x$ $f_{yy} = -2y + 2x$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow$ c.p : $(0,0)$ $Q(X,Y) = 0$

so draw level set at $f(0,0) = 0$:



So it is a saddle pt.

Lagrange Multiplier:

1. $\begin{cases} f(x,y) = xy \\ g(x,y) = x^2 + \frac{1}{4}y^2 = 1 \end{cases}$ for c.p: $\begin{cases} \nabla f = \lambda \cdot \nabla g \\ x^2 + \frac{1}{4}y^2 = 1 \end{cases} \iff \begin{cases} y = \lambda \cdot 2x & ① \\ x = \lambda \cdot \frac{1}{2} y & ② \\ x^2 + \frac{1}{4} y^2 = 1 & ③ \end{cases}$

① → ② $x = \lambda \cdot \frac{1}{2} \cdot \lambda \cdot 2x = \lambda^2 x \Rightarrow (\lambda^2 - 1) \cdot x = 0. \Rightarrow x = 0$ or $\lambda = \pm 1$

if $x=0$ then from ① $y=0$ then ③ does not hold.

if $\lambda = 1$. $y = 2x$ by ①. plug in ③ $x^2 + \frac{1}{4} \cdot 4x^2 = 2x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$

$(\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}})$ $(\frac{-1}{\sqrt{2}}, \frac{-2}{\sqrt{2}})$

if $\lambda = -1$ $y = -2x$ by ① plug in ③ $x = \pm \frac{1}{\sqrt{2}}$ $(\frac{1}{\sqrt{2}}, -\frac{2}{\sqrt{2}})$ $(\frac{-1}{\sqrt{2}}, \frac{2}{\sqrt{2}})$

so there're 4 c.p. The minimal value for $f(x,y) = xy$ is -1 at. $(\frac{\pm 1}{\sqrt{2}}, \frac{\mp 2}{\sqrt{2}})$

2. distance^2: $f(x,y) = (x-2)^2 + (y-1)^2 + (z-4)^2$

$g(x,y) = 2x - y + 3z = 1$

for c.p. $\begin{cases} \nabla f = \lambda \cdot \nabla g \\ 2x - y + 3z = 1 \end{cases} \iff \begin{cases} 2(x-2) = \lambda \cdot 2 & ① \\ 2(y-1) = -\lambda & ② \\ 2(z-4) = 3\lambda & ③ \\ 2x - y + 3z = 1 & ④ \end{cases}$ by ① $x = \lambda + 2$ ② $y = -\frac{\lambda}{2} + 1$

③ $z = \frac{3}{2}\lambda + 4$

plug all in ④

$\Rightarrow \lambda = -2$

c.p: $x = 0$ $y = 2$ $z = 1$. The minimal distance is $\sqrt{(0-2)^2 + (2-1)^2 + (1-4)^2}$

$= \sqrt{14}$.

3. $\frac{1}{2}\cdot$ Surface Area: $f(x,y,z) = xy + yz + xz$ ③

Volume: $g(x,y,z) = \cancel{2}\, xyz = \frac{1}{2}$

for c.p: $\begin{cases} \nabla f = \lambda \nabla g \\ xyz = \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} y+z = \lambda \cdot yz & ① \\ x+z = \lambda \cdot xz & ② \\ x+y = \lambda \cdot xy & ③ \\ xyz = \frac{1}{2} & ④ \end{cases}$

① $\times x \Rightarrow xy + yz = \lambda xyz$

compare with ② $\times y$ ③ $\times z$

we have $xy + yz = xy + yz = xz + yz = \lambda xyz$

$\Rightarrow xy = yz = xz = \frac{1}{2}\lambda xyz = \frac{1}{2}\lambda \cdot \frac{1}{2} = \frac{\lambda}{4}$ plug in ① ② ③ $y+z = x+z = x+y = \frac{\lambda^2}{4}$

$\Rightarrow x = y = z = \frac{\lambda^2}{8}$ plug in ④. $\frac{\lambda^6}{8^3} = \frac{1}{2} \Rightarrow \lambda = 2^{\frac{4}{3}}$ $x = y = z = 2^{-\frac{1}{3}}$

The shape of the box should be. Length $\times$ Wide $\times$ Height $= 2^{-\frac{1}{3}} \times 2^{-\frac{1}{3}} \times 2^{-\frac{1}{3}}$