MATH 234 WORKSHEET 11 22 NOV 2016, TA: JIUYA WANG

Problem 1 : Line integral of Functions
Compute the following line integral:

1. $f(x,y)=xy$, $C:\vec{\gamma}(t)=\begin{pmatrix}t\\t^2\end{pmatrix}$, $0\le t\le 1$
2. $f(x,y,z)=x^2+y^2$, $C:\vec{\gamma}(t)=\begin{pmatrix}3\cos t\\3\sin t\\t\end{pmatrix}$, $0\le t\le 2\pi$
3. Page 142, Ex 1
4. Page 142, Ex 2

1. $\vec{\gamma}'(t)=\begin{pmatrix}1\\2t\end{pmatrix}$ $\|\vec{\gamma}'(t)\|=\sqrt{1+4t^2}$

$\int_0^1 t\cdot t^2\cdot\sqrt{1+4t^2}\,dt$ $\quad\frac{u=1+4t^2}{du=8t\,dt}$

$\int_1^5 \frac{u-1}{4}\cdot\sqrt{u}\cdot\frac{du}{8}=\frac{1}{16}\cdot\left(\frac{5^{5/2}-1}{5}-\frac{5^{3/2}-1}{3}\right)$

2. $\vec{\gamma}'(t)=\begin{pmatrix}-3\sin t\\3\cos t\\1\end{pmatrix}$ $\|\vec{\gamma}'(t)\|=\sqrt{10}$

$\int_0^{2\pi}3^2\cdot\sqrt{10}\,dt=18\sqrt{10}\pi$

3. a) $C:\ \vec{\gamma}(\theta)=\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$ $\quad 0\le\theta\le\frac{\pi}{2}$

distance : $\sqrt{x^2+y^2}$ $\quad\vec{\gamma}'(\theta)=\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}$

$\int_0^{\frac{\pi}{2}}\sqrt{\cos^2\theta+\sin^2\theta}\cdot 1\cdot d\theta=\frac{\pi}{2}$

Arc Length: $\int_0^{\frac{\pi}{2}}\|\vec{\gamma}'(\theta)\|d\theta=\frac{\pi}{2}$ $\quad$ Ave Distance $=\frac{\pi/2}{\pi/2}=1$

b) $\int_0^{\frac{\pi}{2}}\theta\cdot\|\vec{\gamma}'(\theta)\|d\theta=\frac{\pi^2}{4}\cdot\frac{1}{2}$

Ave Polar $=\frac{\pi^2/8}{\pi/2}=\frac{\pi}{4}$

4. a) $\vec{\gamma}_1(t)=\begin{pmatrix}1\\t\end{pmatrix}$ $\quad 0\le t\le a$

$\vec{\gamma}_2(t)=\begin{pmatrix}1-t\\a\end{pmatrix}$ $\quad 0\le t\le 1$

Arc Length : $a+1$

$\int_0^a 1\cdot\|\vec{\gamma}_1'(t)\|\,dt=a$

$\int_0^1(1-t)\|\vec{\gamma}_2'(t)\|\,dt=\frac{1}{2}$

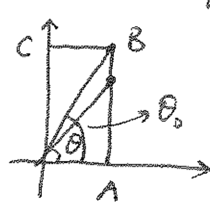
Ave x : $\frac{a+1/2}{a+1}$

$\int_0^a t\|\vec{\gamma}_1'(t)\|\,dt=\frac{a^2}{2}$

$\int_0^1 a\cdot\|\vec{\gamma}_2'(t)\|\,dt=a$

Ave y: $\frac{a+a^2/2}{a+1}$

b)



$\tan\theta=\frac{y}{1}=y$ for $0\le\theta\le\theta_0$

$\tan\theta=\frac{a}{x}$ for $\theta>\theta_0$

Similarly, for $\vec{\gamma}_2$

$\cancel{\int_0^1\arctan\frac{a}{1-t}\,dt}$

Next Page Bottom →

Problem 2 : Line integral of Vector Fields
Compute the following line integral of vector fields:

1. $\vec{F}=\begin{pmatrix}x+y\\2y\end{pmatrix}$, $C:\vec{\gamma}(t)=(t,t^2)$, $0\le t\le 1$ from $t=0$ to $t=1$
2. $\vec{F}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, $C:\vec{\gamma}(t)=\begin{pmatrix}3\cos t\\3\sin t\\t\end{pmatrix}$, $0\le t\le 2\pi$ from $t=0$ to $t=2\pi$

1. $\int_0^1\begin{pmatrix}t+t^2\\2t^2\end{pmatrix}\cdot\begin{pmatrix}1\\2t\end{pmatrix}dt=\int_0^1(t+t^2+4t^3)\,dt$

$=\frac{1}{2}+\frac{1}{3}+1$

2. $\int_0^{2\pi}\begin{pmatrix}3\cos t\\3\sin t\\t\end{pmatrix}\cdot\begin{pmatrix}-3\sin t\\3\cos t\\1\end{pmatrix}dt=\int_0^{2\pi}t\,dt=2\pi^2$

$\int_0^a\arctan t\cdot\|\vec{\gamma}_1'(t)\|\,dt=\int_0^a\arctan t\,dt$

$\tan u=t$ $\quad\int_0^{\arctan a}u\cdot\sec^2u\,du=(u\cdot\tan u+\ln|\cos u|)\Big|_0^{\arctan a}$

$=a\cdot\arctan a+\ln\frac{1}{\sqrt{1+a^2}}$

Problem 3 : Conservative Vector Field

1. Given $\vec{F} = \begin{pmatrix} 2xe^{xy} + x^2ye^{xy} \\ x^3e^{xy} + 2y \end{pmatrix}$, is $\vec{F}$ conservative or not? If so, find the potential function. ($=P$, $=Q$)

2. Given $\vec{F} = \begin{pmatrix} y \\ z \\ x \end{pmatrix}$, is $\vec{F}$ conservative or not? ($=P$, $=Q$, $=R$)

3. Consider the vector field in 3.1, C is the upper half unit circle starting from $(-1, 0)$ to $(1, 0)$, compute the line integral of vector field.

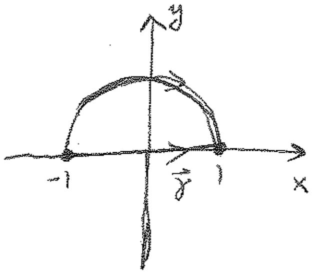
1. $P_y = 2x\cdot e^{xy}\cdot x + x^2\cdot e^{xy} + x^2\cdot y\cdot e^{xy}\cdot x$

$Q_x = 3x^2\cdot e^{xy} + x^3\cdot e^{xy}\cdot y + 0$

$P_y = Q_x$ so $\vec{F}$ is conservative.

2. $P_y = 1 \neq Q_x$ no

3.



Using $\vec{r}(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$ $-1 \leq t \leq 1$

$$\int_{-1}^{1} \begin{pmatrix} 2t \\ t^3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt$$

$$= \int_{-1}^{1} (2t)\, dt = t^2 \Big|_{-1}^{1} = 0$$

Problem 1.4.

b). For $\vec{r}_2(t)$,

$\tan(\frac{\pi}{2} - \theta) = \frac{x}{a}$

$$\int_0^1 \left(\frac{\pi}{2} - \arctan\frac{1-t}{a}\right) dt$$

$\tan u = \frac{1-t}{a}$

$$\frac{\pi}{2} - \int_{\arctan\frac{1}{a}}^{0} (-a)\, u \cdot (\tan u)'\, du$$

$$= \frac{\pi}{2} - a\int_0^{\frac{\pi}{2} - \arctan a} u \cdot \sec^2 u\, du$$

$$= \frac{\pi}{2} - a\cdot(u\cdot\tan u + \ln|\cos u|)\Big|_0^{\frac{\pi}{2} - \arctan a}$$

$$a\cdot\arctan a + \ln\frac{1}{\sqrt{1+a^2}} + \frac{\pi}{2} - \arctan\frac{1}{a} - a\ln\frac{a}{\sqrt{1+a^2}}$$

$$= (a+1)\arctan a - a\ln a - (a-1)\cdot\ln\frac{1}{\sqrt{a^2+1}}$$

Are Polar: $\arctan a - \frac{a\ln a}{a+1} + \frac{a-1}{a+1}\left(\ln\sqrt{a^2+1}\right)$

(Check the algebra. I'm not 100% sure, but the integral should follow as above)