MATH 234 WORKSHEET 12 29 Nov 2016, TA: JIUYA WANG

Problem 1 : Flux Integral

Compute the following line integral:

☆ Notice in the discussion, we forgot to add in $\|\vec\gamma'(t)\|$ in integrand!!

1. $\vec v = \begin{pmatrix} x+y \\ 2y \end{pmatrix}$, $C : \vec\gamma(t) = (t, t^2), 0 \le t \le 1$, $\vec N$ the upward normal

2. $\vec v = \begin{pmatrix} xy^2 \\ x^2y \end{pmatrix}$, C: unit circle, $\vec N$ the outward normal

1. $\vec T = \vec\gamma'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$ $\quad \vec N = \begin{pmatrix} -2t \\ 1 \end{pmatrix} \frac{1}{\sqrt{1+4t^2}}$

$$\int_0^1 \vec v \cdot \vec N \, ds = \int_0^1 \begin{pmatrix} t+t^2 \\ 2t^2 \end{pmatrix} \cdot \begin{pmatrix} -2t \\ 1 \end{pmatrix} \frac{1}{\sqrt{1+4t^2}} \cdot \|\vec\gamma'(t)\| dt$$

$$= \int_0^1 (\cancel{-2t^2} - 2t^3 + \cancel{2t^2})\, dt = -\frac{t^4}{2}\Big|_0^1 = -\frac{1}{2}$$

2. $\vec\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ $\quad 0 \le t \le 2\pi$

$\vec\gamma'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$

$\vec N(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

(flip x and y part, outward means choosing $\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ instead of $\begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix}$)

$\vec v \cdot \vec N = \begin{pmatrix} \cos t \cdot \sin^2 t \\ \cos^2 t \cdot \sin t \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$\|\vec\gamma'(t)\| = 1$

$$\int_0^{2\pi} 2\cos^2 t \sin^2 t \cdot 1 \cdot dt = \int_0^{2\pi} \frac{1}{2}\sin^2 2t \, dt$$

$$= \int_0^{2\pi} \frac{1}{2} \cdot \frac{1-\cos 4t}{2} dt$$

$$= \frac{1}{4} t \Big|_0^{2\pi} = \frac{\pi}{2}$$

Problem 2 : More about Conservative Field

1. Given $\vec F = \begin{pmatrix} 6xy + 4e^y \\ 3x^2 + 4xe^y \end{pmatrix}\begin{matrix} =P \\ =Q \end{matrix}$, is $\vec F$ conservative or not?

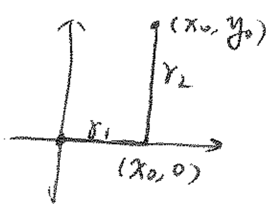
2. Suppose $\vec F$ is conservative, find the function f such that $\vec\nabla f = \vec F$.

1. By Clairaut's Thm, $P_y = 6x + 4e^y = Q_x = 6x + 4e^y$

$\vec F$ is conservative

2. Assume $f(0,0) = 0$. $\quad \int_{(0,0)}^{(x_0,y_0)} \vec F \cdot d\vec s = f(x_0,y_0) - f(0,0) = f(x_0,y_0)$

Choosing path $\vec\gamma_1(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$ $\quad 0 \le t \le x_0$ $\quad \vec\gamma_2(t) = \begin{pmatrix} x_0 \\ t \end{pmatrix}$ $\quad 0 \le t \le y_0$



$$\int_{\gamma_1} \vec F \cdot d\vec s = \int_0^{x_0} \begin{pmatrix} 4 \\ 3t^2 + 4t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot dt = 4x_0$$

$$\int_{\gamma_2} \vec F \cdot d\vec s = \int_0^{y_0} \begin{pmatrix} 6x_0 t + 4e^t \\ 3x_0^2 + 4x_0 e^t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt = \int_0^{y_0} (3x_0^2 + 4x_0 e^t) dt$$

$$= 3x_0^2 \cdot y_0 + 4x_0(e^{y_0} - 1)$$

$f(x_0, y_0) = 4x_0 + 3x_0^2 y_0 + 4x_0 e^{y_0} - 4x_0 = 3x_0^2 y_0 + 4x_0 e^{y_0}$

So $f(x,y) = 3x^2 y + 4xe^y$