Problem 1 : Green's Theorem

Compute the following line integral in two ways: by definitions and by Green's Thm:

1. Page 159 5.a
2. 5. c
3. 5. k
4. 5. l

1. 1) $\vec{F} = \begin{pmatrix} xy \\ xy \end{pmatrix}$

$\vec{\gamma}_1 = \begin{pmatrix} t \\ 0 \end{pmatrix}$ $0 \le t \le 1$

$\vec{\gamma}_2 = \begin{pmatrix} 1 \\ t \end{pmatrix}$ $0 \le t \le 1$

$\vec{\gamma}_3 = \begin{pmatrix} 1-t \\ 1 \end{pmatrix}$ $0 \le t \le 1$

$\vec{\gamma}_4 = \begin{pmatrix} 0 \\ 1-t \end{pmatrix}$ $0 \le t \le 1$

For $\vec{\gamma}_1$: $\int_0^1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \vec{\gamma}_1'(t)\,dt = 0$

$\vec{\gamma}_2$: $\int_0^1 \begin{pmatrix} t \\ t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt = \frac{1}{2}$

$\vec{\gamma}_3$: $\int_0^1 \begin{pmatrix} 1-t \\ 1-t \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} dt = -\frac{1}{2}$

$\vec{\gamma}_4$: 0

So $\oint_{\partial D} \vec{F} \cdot d\vec{s} = 0$

2). by Green $Q_x - P_y = y - x$.

$\iint_D (y-x)\,dA = \int_0^1 \int_0^1 (y-x)\,dx\,dy$

$= \int_0^1 (yx - \frac{x^2}{2})\Big|_0^1 dy$

$= \int_0^1 (y - \frac{1}{2})\,dy = 0$

2. 1) $\vec{F} = \begin{pmatrix} y\cos x \\ y \sin x \end{pmatrix}$ $\vec{\gamma}_1 = \begin{pmatrix} \frac{\pi}{2} t \\ 1 \end{pmatrix}$ $0 \le t \le 1$

$\vec{\gamma}_2 = \begin{pmatrix} \frac{\pi}{2} \\ 1+t \end{pmatrix}$ $0 \le t \le 1$ $\vec{\gamma}_3 = \begin{pmatrix} \frac{\pi}{2}\cdot(1-t) \\ 2 \end{pmatrix}$ $0 \le t \le 1$

$\vec{\gamma}_4 = \begin{pmatrix} 0 \\ 2-t \end{pmatrix}$ $0 \le t \le 1$

For $\vec{\gamma}_2$ $\int_0^1 \begin{pmatrix} 0 \\ 1+t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt = \frac{3}{2}$

Problem 2 : Surface Integral Part 1: Area, Mass

Given the unit sphere $\vec{x}(\theta,\phi) = \begin{pmatrix} \cos\phi \sin\theta \\ \sin\phi \sin\theta \\ \cos\theta \end{pmatrix}$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$,

1. What is the normal vector $\vec{x}_\theta \times \vec{x}_\phi$?
2. What is the area by surface integral?
3. What is the unit normal?
4. Suppose the density function $\mu(\theta,\phi) = \cos^2\theta$, compute the total mass of the this unit sphere?

$\vec{\gamma}_3$: $\int_0^1 \begin{pmatrix} 2\cos[(1-t)\cdot\frac{\pi}{2}] \\ 2\sin[(1-t)\frac{\pi}{2}] \end{pmatrix} \begin{pmatrix} -\frac{\pi}{2} \\ 0 \end{pmatrix} dt$

$= -\pi \int_0^1 \cos[\frac{\pi}{2}(1-t)]\,dt$

$u = \frac{\pi}{2}(1-t)$, $du = -\frac{\pi}{2}dt$: $-\pi \cdot \int_{\frac{\pi}{2}}^0 \cos u \cdot (-\frac{2}{\pi})\,du = -2$

$\vec{\gamma}_4$: $\int_0^1 \begin{pmatrix} 2-t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dt = 0$

$\vec{\gamma}_1$: $\int_0^1 \begin{pmatrix} \cos\frac{\pi}{2}t \\ \sin\frac{\pi}{2}t \end{pmatrix} \cdot \begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix} dt$

$= \int_0^1 \cos\frac{\pi}{2}t \cdot \frac{\pi}{2}\,dt = 1$

So. $\oint_{\partial D} \vec{F} \cdot d\vec{s} = \frac{1}{2}$

2) $Q_x - P_y = y\cdot\cos x - \cos x$

$\int_0^{\frac{\pi}{2}} \int_1^2 (y\cdot\cos x - \cos x)\,dy\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}\cdot\cos x\,dx = \frac{1}{2}$

3. 1) $\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad 0 \le t \le 2\pi$

$\vec{F} = \begin{pmatrix} x^2 y \\ -xy^2 \end{pmatrix}$

$\oint_{\partial D} \cdot \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \begin{pmatrix} \cos^2 t \cdot \sin t \\ -\cos t \cdot \sin^2 t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt$

$= \int_0^{2\pi} -2\sin^2 t \cos^2 t \, dt$

$-2\sin^2 t \cdot \cos^2 t = \frac{\sin^2 2t}{-2} = \frac{1}{-2} \cdot \frac{1-\cos 4t}{2}$

$\int_0^{2\pi} -2\sin^2 t \cos^2 t \, dt = -\frac{1}{4} \cdot \int_0^{2\pi} (1-\cos 4t) \, dt$

$= -\frac{\pi}{2}$

2). $Q_x - P_y = -y^2 - x^2$

$\iint_D -(y^2+x^2) \, dA = \int_0^{2\pi} \int_0^1 (-r^2) \cdot r \, dr \, d\theta$

$= -\frac{1}{4} \cdot 2\pi = -\frac{\pi}{2}$

4. 1) $\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \qquad \vec{r}'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$

$\vec{N} = \begin{pmatrix} \cos t \\ -(-\sin t) \end{pmatrix}$ or $\begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix}$

we need outward direction, so

$\vec{N} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$\cancel{\oint_{\partial D} \vec{F} \cdot d\vec{s} = \int_0^{2\pi}}$

$\oint_{\partial D} \vec{v} \cdot \vec{N} \, ds = \int_0^{2\pi} \begin{pmatrix} \cos t \cdot \sin^2 t \\ \cos^2 t \cdot \sin t \end{pmatrix} \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \cdot \|\vec{r}'(t)\| dt$

$= \int_0^{2\pi} 2\cos^2 t \sin^2 t \, dt = 2\left(-\frac{\pi}{2(-2)}\right) = \frac{\pi}{2}$

2). $P_x + Q_y = y^2 + x^2$

$\iint_D (x^2+y^2) \, dA = \frac{\pi}{2}$