1\. 1) $\vec{x}_\theta = \begin{pmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ -\sin\theta \end{pmatrix}$ $\quad\quad$ $\vec{x}_\phi = \begin{pmatrix} -\sin\phi\,\sin\theta \\ \cos\phi\,\sin\theta \\ 0 \end{pmatrix}$

$$\vec{x}_\theta \times \vec{x}_\phi = \begin{vmatrix} i & j & k \\ \cos\phi\cos\theta & \sin\phi\cos\theta & -\sin\theta \\ -\sin\phi\sin\theta & \cos\phi\sin\theta & 0 \end{vmatrix} = \begin{pmatrix} \sin^2\theta\cos\phi \\ \sin^2\theta\sin\phi \\ \cos\theta\sin\theta \end{pmatrix}$$

2)

$$\|\vec{x}_\theta \times \vec{x}_\phi\| = \sqrt{\sin^4\theta\cos^2\phi + \sin^4\theta\sin^2\phi + \cos^2\theta\sin^2\theta} = \sin\theta$$

3) $$\text{Area} = \int_0^{\pi}\int_0^{2\pi} \sin\theta \cdot d\phi\, d\theta = 2\pi\cdot 2 = 4\pi$$

unit normal $$\vec{N} = \frac{\vec{x}_\theta\times\vec{x}_\phi}{\|\vec{x}_\theta\times\vec{x}_\phi\|} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

4). $$\int_0^{\pi}\int_0^{2\pi} \cos^2\theta\cdot\sin\theta\, d\phi\, d\theta = 2\pi\cdot\int_0^{\pi}\cos^2\theta\sin\theta\, d\theta = 2\pi\int_1^{-1} -u^2\cdot du = \frac{4}{3}\pi$$

5). $$\int_0^{\pi/2}\int_0^{2\pi} \cos^3\theta\cdot\sin\theta\, d\phi\, d\theta = 2\pi\cdot\int_0^{\pi/2}\cos^3\theta\sin\theta\, d\theta = 2\pi\int_1^{0} -u^3\, du = \frac{\pi}{2}$$

$$\int_0^{\pi/2}\int_0^{2\pi} \cos^2\theta\sin\theta\, d\phi\, d\theta = \frac{2}{3}\pi.$$

$$\bar{z} = \frac{\frac{\pi}{2}}{\frac{2}{3}\pi} = \frac{3}{4}$$

2\. 1). $$\int_0^{\pi}\int_0^{2\pi} \vec{N}\cdot(\vec{x}_\theta\times\vec{x}_\phi)\, d\phi\, d\theta = \int_0^{\pi}\int_0^{2\pi}\sin\theta\cdot d\phi\, d\theta = 4\pi$$

$$\begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}\cdot\begin{pmatrix} \sin^2\theta\cos\phi \\ \sin^2\theta\sin\phi \\ \cos\theta\sin\theta \end{pmatrix} = \sin\theta$$

2)
$$\begin{pmatrix} \sin\phi \sin\theta \\ -\cos\phi \sin\theta \\ \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \sin^2\theta \cos\phi \\ \sin^2\theta \sin\phi \\ \cos\theta \sin\theta \end{pmatrix} = \cos^2\theta \sin\theta$$

$$\int_0^{\pi} \int_0^{2\pi} \vec{v} \cdot (\vec{X}_\theta \times \vec{X}_\phi) \, d\phi \, d\theta = \int_0^{\pi} \int_0^{2\pi} \cos^2\theta \sin\theta \, d\phi \, d\theta = \frac{4}{3}\pi$$

3. 1). $\operatorname{div}(\vec{v}) = 3$ $\quad \iiint_V 3 \, dV = 3 \cdot \operatorname{Vol}(V) = 4\pi$

V: unit ball

2) $\operatorname{div}(\vec{v}) = 1$ $\quad \iiint_V 1 \, dV = 1 \cdot \operatorname{Vol}(V) = \frac{4}{3}\pi$

3) $\operatorname{div}(\vec{v}) = 0$. so flux is 0

4. 1) Choose the unit disk: $x^2+y^2 \leq 1$ as the domain (instead of upper sphere). then it is Green's Thm actually.

$$\vec{X}(u,v) = \begin{pmatrix} u\cos v \\ u \sin v \\ 0 \end{pmatrix} \qquad \vec{X}_u \times \vec{X}_v = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}$$

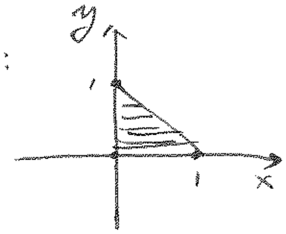
$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix} \qquad \int_0^1 \int_0^{2\pi} 2cu \, dv \, du = 2c \cdot 2\pi \cdot \frac{1}{2} = 2c\pi$$

2) The plane is $x+y+z=1$. $\quad \vec{x}(x,y) = \begin{pmatrix} x \\ y \\ 1-x-y \end{pmatrix}$

$$\vec{X}_x \times \vec{X}_y = \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \int_0^1 \int_0^{1-y} -2 \, dx \, dy = -1$$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{pmatrix} -2z \\ -2x \\ -2y \end{pmatrix}$$

Projection:



$$\operatorname{curl} \vec{F} \cdot (\vec{X}_x \times \vec{X}_y) = -2(x+y+z) = -2$$