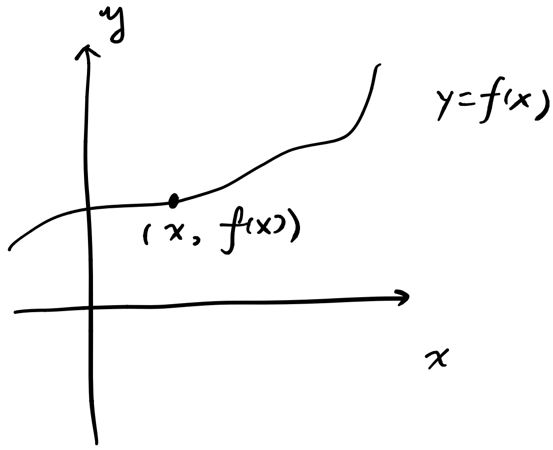
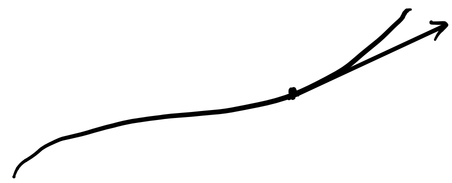
Curves and Surfaces

Curves



$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$



- tangent vector (velocity)

$$\vec{r}'(t) = \frac{d\vec{r}(t)}{dt} = \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}$$

- acceleration

$$\vec{r}''(t) = \frac{d\vec{r}'(t)}{dt} = \begin{pmatrix} x''(t) \\ y''(t) \\ z''(t) \end{pmatrix}$$

- arc length

$\|\vec{r}'(t)\|$: scalar velocity

$$\int_a^t \|\vec{r}(u)\| \cdot du = s(t)$$

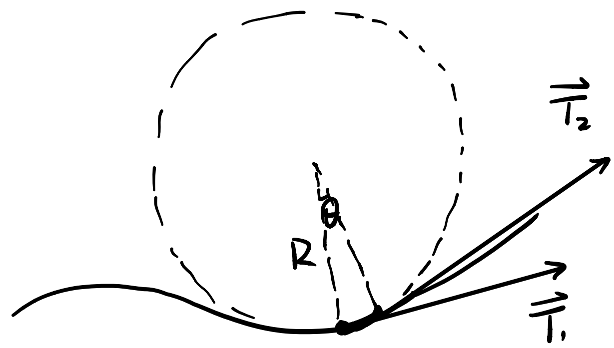
- unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

- principal unit normal vector $\vec{N}$

curvature κ

$$\frac{d\vec{T}(t(s))}{ds} = \kappa \cdot \vec{N}$$



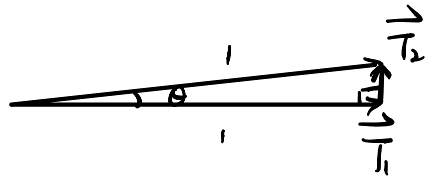
$$\kappa = \frac{1}{R}$$

$$\vec{N} \perp \vec{T}$$

$$\frac{dt}{ds} \cdot \frac{ds}{dt} = 1$$

$$\frac{d\vec{T}(t(s))}{ds} = \frac{d\vec{T}}{dt} \cdot \frac{dt}{ds}$$

$$= \frac{d\vec{T}}{dt} \cdot \frac{1}{\|\vec{r}'(t)\|}$$



$$\vec{T}_1 - \vec{T}_2 = \Delta\vec{T}$$

$$\|\Delta\vec{T}\| \sim \theta \cdot 1 \sim \theta$$

$$\Delta\vec{T} \perp \vec{T}$$

$$\Delta s \sim \theta \cdot R$$

Recall Chain Rule

$$\frac{d f(g(x))}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

$$f'(g(x)) \cdot g'(x)$$

Example. $\vec{\gamma}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$

1). tangent vector ? Acceleration ?

2) Compute the arc length from $t=0$ to $t=1$.

3) unit tangent vector ?

4) principal unit normal vector ? curvature ?

1) $\vec{\gamma}'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$ $\vec{\gamma}''(t) = \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix}$

2) $\|\vec{\gamma}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1^2} = \sqrt{2}$

$\int_0^1 \sqrt{2} \cdot dt = \sqrt{2}$

3) $\vec{T}(t) = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$

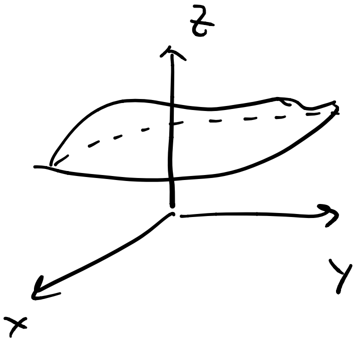
4) $\frac{d\vec{T}}{ds} = \frac{d\vec{T}(t)}{dt} \cdot \frac{1}{\|\vec{\gamma}'(t)\|} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$

$= \frac{1}{2} \cdot \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix} = \kappa \cdot \vec{N}$

$\kappa = \frac{1}{2}$. $\vec{N} = \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix}$

Surfaces

$z = f(x, y)$

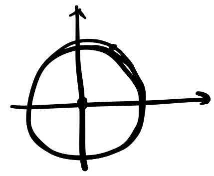


paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$a = b = c = 1$ sphere.

hyperboloid. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

$x^2 + y^2 = 1$



elliptic cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$