Derivatives of Multi-variable Function.

- Partial Derivative $\quad f = f(x_1, \cdots, x_n)$

$$\frac{\partial f}{\partial x_i} = \lim_{\varepsilon \to 0} \frac{f(x_1, \cdots, x_i+\varepsilon, \cdots, x_n) - f(x_1, \cdots, x_n)}{\varepsilon}$$

eg. $f(x,y) = x^2 + y^2 \qquad \frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 2y$.

- Differential

$$df = \sum_i \frac{\partial f}{\partial x_i} \cdot dx_i \qquad df = 2x \cdot dx + 2y \cdot dy$$

at (x_0, y_0, z_0)

Idea: linear approximation of f. $\qquad \Delta z = 2x_0 \Delta x + 2y_0 \Delta y$

Questions following this line: $\quad z - z_0 = 2x_0 \cdot (x - x_0) + 2y_0 (y - y_0)$

(*) Tangent Plane

eg. $z = f(x,y) = x \cdot y \qquad$ Q: Tangent Plane at $(1,1,1)$?

$$f_x = y \qquad f_y = x \qquad df\Big|_{(1,1,1)} = y\Big|_{(1,1,1)} dx + x\Big|_{(1,1,1)} dy = 1 \cdot dx + 1 \cdot dy$$

$$z - 1 = 1 \cdot (x-1) + 1 \cdot (y-1)$$

(*) Normal Vector of Tangent Plane.

$$f_x(x_0,y_0) \cdot (x - x_0) + f_y(x_0,y_0) \cdot (y - y_0) - (z - z_0) = 0$$

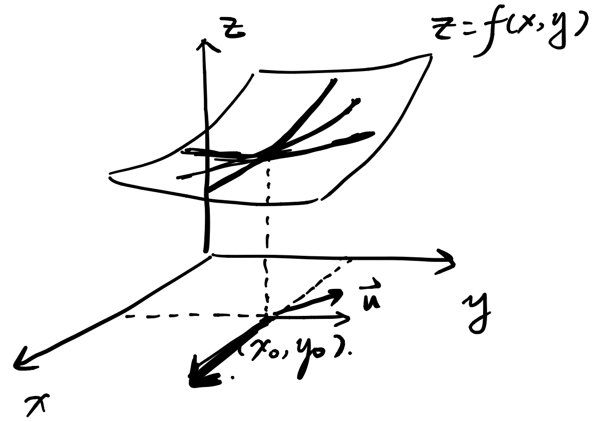
$$\begin{pmatrix} f_x(x_0, y_0) \\ f_y(x_0, y_0) \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0 \qquad \vec{N} = \begin{pmatrix} f_x(x_0, y_0) \\ f_y(x_0, y_0) \\ -1 \end{pmatrix}$$

(*) Gradient $\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$ for $f = f(x,y)$

if $f = f(x_1, \cdots, x_n)$ then $\nabla f = \begin{pmatrix} f_{x_1}(\vec{x}) \\ \vdots \\ f_{x_n}(\vec{x}) \end{pmatrix}$

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i = \nabla f \cdot \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix} = \nabla f \cdot d\vec{x}$$

(*) Directional Derivative



$\vec{u}$: unit vector $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$$D_{\vec{u}} f = \lim_{\varepsilon \to 0} \frac{f(x+u_1\varepsilon, y+u_2\varepsilon) - f(x,y)}{\varepsilon}$$

$$= \nabla f \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \nabla f \cdot \vec{u}$$

eg $\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\nabla f \cdot \vec{u} = f_x$

(*) Property of ∇f: ∇f is the direction where f increases the fastest.

since $\nabla f \cdot \vec{u} = \|\nabla f\| \cdot \cos\theta$ attains the maximal value when $\cos\theta = 1 \Leftrightarrow \theta = 0$

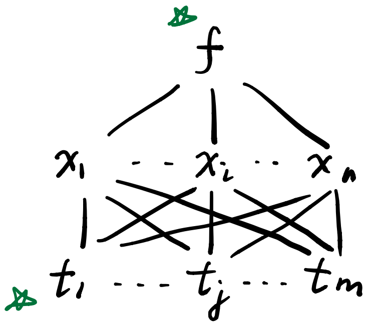
eg. $f = x^2 + y^2$ then at $(1,1,2)$ what is the direction where f increases the fastest?

$$\nabla f \Big|_{(1,1,2)} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}\Big|_{(1,1,2)} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$-\nabla f = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$ is the direction where f decreases the fastest. $\vec{u} \longleftarrow \overset{\pi}{\bullet} \longrightarrow \nabla f$

- Chain Rule $\quad f = f(x_1, \cdots, x_n)$

$x_i = x_i(t_1, \cdots, t_m)$ then



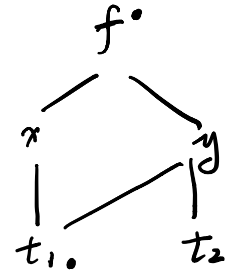
$$\frac{\partial f}{\partial t_1} = \sum_i \frac{\partial f}{\partial x_i} \cdot \frac{\partial x_i}{\partial t_1}$$

eg $\quad f(x, y) = x^2 + y^2 \qquad x = x(t_1, t_2) = 2t_1$

$y = y(t_1, t_2) = t_2^2 + t_1^2$

$$\frac{\partial f}{\partial t_1} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t_1} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t_1} = 2x \cdot 2 + 2y \cdot 2t_1$$

$$\frac{\partial f}{\partial t_2} = 2y \cdot 2t_2$$



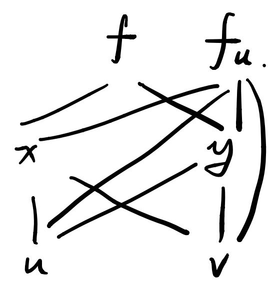
- Higher Order Derivatives.

$$\frac{\partial f_x(x,y)}{\partial x} = f_{xx}(x,y) \qquad \frac{\partial f_x(x,y)}{\partial y} = f_{xy}(x,y)$$

Similarly for $f_{yx}, f_{yy}, f_{xxy}, \cdots$

eg. $f(x,y) = x^2 + y^2 \qquad x(u,v) = u^2 + v^2$

$y(u,v) = 2 \cdot u \cdot v$



$f_{uu} = ?$

$$f_u = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= 2x \cdot 2u + 2y \cdot 2v = f_u(x, y, u, v)$$

$$f_{uu} = \frac{\partial f_u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f_u}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial f_u}{\partial u}$$

$$= 4u \cdot 2u + 4v \cdot 2v + 4x$$

- Application of Derivatives
  - * Local Min/Max $\quad f_x = f_y = 0$
  - * Global Min/Max (within Region).
    - $f_x = f_y = 0$
    - points on the boundary
    - points where f_x or f_y is not defined.