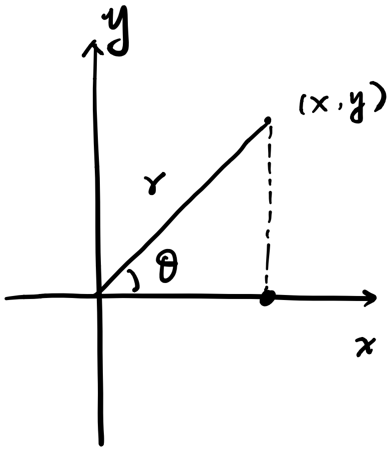
Polar Coordinates / Cylinder Coordinates / Sphere Coordinates

2D



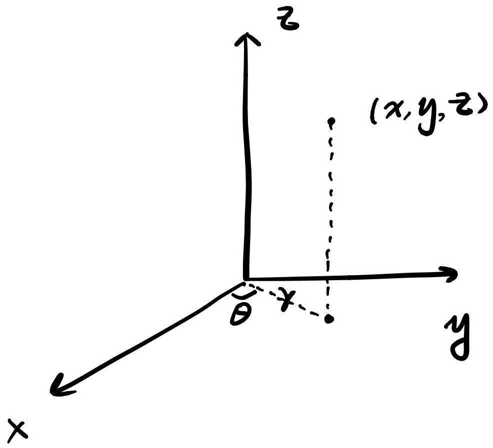
$$\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \end{cases}$$

$r \geq 0$

$0 \leq \theta < 2\pi$

$|\det(J)| = r$

3D



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

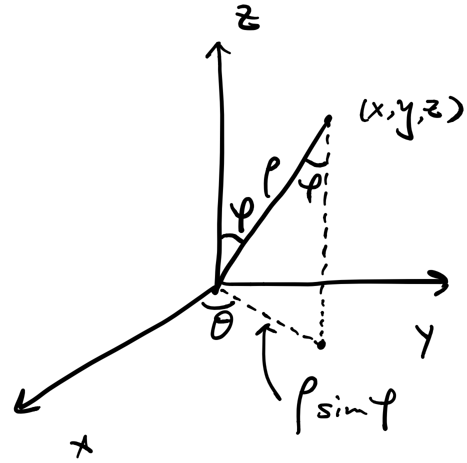
$r \geq 0$

$0 \leq \theta < 2\pi$

$z \in \mathbb{R}$

$|\det(J)| = r$

3D



$x = \rho\cdot\sin\varphi\cdot\cos\theta$

$y = \rho\cdot\sin\varphi\cdot\sin\theta$

$z = \rho\cdot\cos\varphi$

$\rho \geq 0$

$0 \leq \theta < 2\pi$

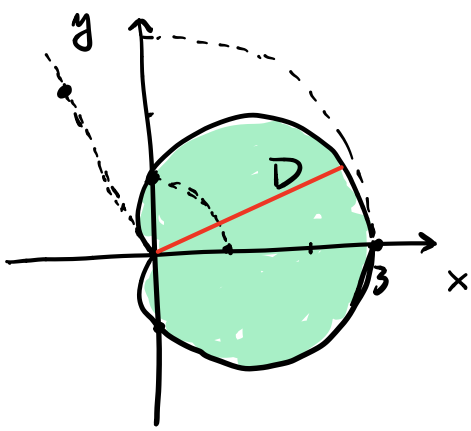
$0 \leq \varphi \leq \pi$

$|\det(J)| = \underline{\rho^2\cdot\sin\varphi}$

eg. 2D. Compute the area of the region bounded by $r = 1 + 2\cdot\cos\theta$ on x-y plane.

$\theta = \frac{2\pi}{3}$

$r = 0$



$$\iint_D 1\cdot dA = \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}\int_0^{1+2\cos\theta} 1\cdot \underline{r}\, dr\, d\theta$$

$$= \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} \left.\frac{r^2}{2}\right|_0^{1+2\cos\theta} d\theta$$

$$= \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} \frac{1 + 4\cos^2\theta + 4\cos\theta}{2}\, d\theta$$

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

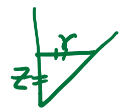
$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

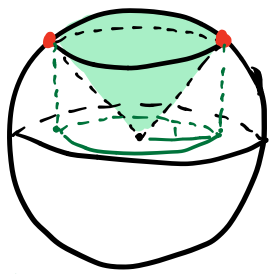
3D. Compute the volume of the region bounded by

$x^2+y^2+z^2=2$ and.

$z=\sqrt{x^2+y^2}$

$z=r$.





$$\iiint_D 1 \cdot dV$$

in cylinder coordinate.

$$= \int_0^{2\pi}\int_0^{1}\int_r^{\sqrt{2-r^2}} 1 \cdot \underline{\underline{r}}\; dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1} r\cdot\left(\sqrt{2-r^2} - r\right)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1} r\cdot\sqrt{2-r^2}\,dr\,d\theta \; - \int_0^{2\pi}\int_0^{1} r^2\,dr\,d\theta$$

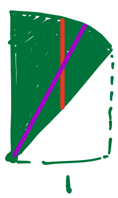
$$= \overset{=2\pi.}{\left(\int_0^{2\pi} 1\,d\theta\right)}\cdot\left(\int_0^{1} r\cdot\sqrt{2-r^2}\,dr\right)^{??} - \underset{=2\pi}{\left(\int_0^{2\pi} 1\,d\theta\right)}\cdot\left(\int_0^{1} r^2\,dr\right) = \frac{1}{3}$$

$u = 2-r^2$

$du = -2r\,dr$

$$= 2\pi\cdot\left(-\int_2^{1} \sqrt{u}\cdot\frac{du}{2}\right) - 2\pi\cdot\frac{1}{3}$$

$$= \frac{2\pi}{2}\cdot\left.\frac{u^{\frac{3}{2}}}{3/2}\right|_1^2 - 2\pi\cdot\frac{1}{3}$$



$$\int_0^1 \sqrt{1-r^2}\cdot dr \overset{r=\cos\theta}{=}$$

$dr = -\sin\theta\,d\theta$

$$\int_{\frac{\pi}{2}}^{0} -\sin^2\theta\, d\theta =$$

in sphere coordinate.

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}} \rho^2\cdot\sin\varphi \; d\rho\,d\varphi\,d\theta$$

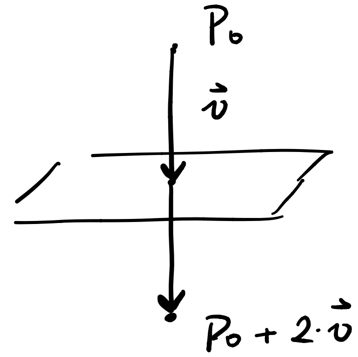
$$= \left(\int_0^{2\pi} 1\,d\theta\right)\cdot\left(\int_0^{\frac{\pi}{4}} \sin\varphi\,d\varphi\right)\cdot\left(\int_0^{\sqrt{2}} \rho^2\,d\rho\right)$$

$$= 2\pi\cdot\left(1-\frac{\sqrt{2}}{2}\right)\cdot\frac{\sqrt{2}^{\,3}}{3}$$

$f(x,y,z) = ax+by+cz+d.$

$f(x,y,z)=0$

$P_0 : (x_0, y_0, z_0)$



$$\vec{v} = \underbrace{(a, b, c)}_{= \vec{n}} \cdot k$$

$\vec{P_0} + \vec{v}$ satisfies the plane.

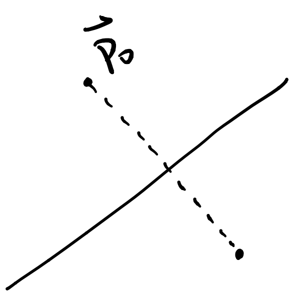
$$\vec{P_0} + \vec{v} = \vec{P_0} + k\cdot\vec{n}$$

$$a\cdot(x_0 + k\cdot a) + b\cdot(y_0 + k\cdot b) + c\cdot(z_0 + k\cdot c) + d = 0.$$

$$k = -\frac{ax_0 + by_0 + cz_0 + d}{a^2+b^2+c^2} = -\frac{f(x_0, y_0, z_0)}{\|\vec{n}\|^2}$$

$\Downarrow$ plug in k.

$$\vec{Q} = \vec{P_0} + 2k\vec{n} = \vec{P_0} - 2\cdot\frac{f(x_0, y_0, z_0)}{\|\vec{n}\|^2}\cdot\vec{n}$$



$f(x,y) = ax+by+c$ $\qquad f(x,y)=0.$

$$k = -\frac{f(x_0, y_0)}{\|\vec{n}\|^2}$$

$$\vec{Q} = \vec{P_0} + 2k\vec{n}$$