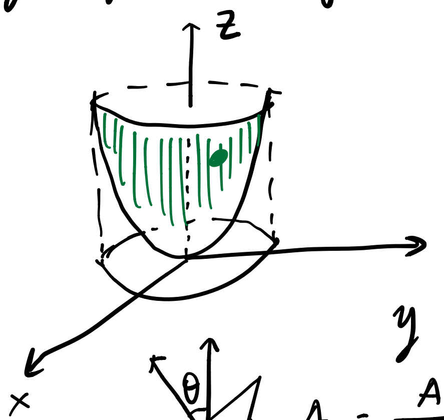
Surface Area / Line Integral.

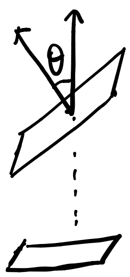
$$\vec{r}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} \qquad \vec{r}_u = \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} \qquad \vec{r}_v = \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix}$$

eg. if $z = x^2+y^2 = f(x,y)$ $\vec{r}(x,y) = \begin{pmatrix} x \\ y \\ x^2+y^2 \end{pmatrix}$ gives a surface in $\mathbb{R}^3$.

$$\vec{r}_x = \begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix} \quad \vec{r}_y = \begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix}$$



Q: What is the area of this surface with $z \le 4$?



$A_1 = \dfrac{A_2}{|\cos\theta|}$ ← we get an extra factor when we project to xy-plane.

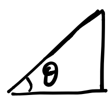
A_2

$$\vec{N} = \begin{pmatrix} f_x \\ f_y \\ -1 \end{pmatrix}$$

Recall from differential

$$\underset{\Delta z}{df} = f_x \cdot \underset{\Delta x}{dx} + f_y \cdot \underset{\Delta y}{dy}.$$

$$f_x \cdot \Delta x + f_y \Delta y - \Delta z = 0$$



$$\cos\theta = \frac{\vec{N}\cdot(0,0,1)}{\|\vec{N}\|\cdot 1} = \frac{-1}{\sqrt{f_x^2+f_y^2+1^2}}$$

Area: $$\iint_{D:\, x^2+y^2\le 4} \sqrt{1+f_x^2+f_y^2}\cdot dA = \iint_D \sqrt{1+4x^2+4y^2}\cdot dA.$$

$$= \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\cdot r\cdot dr\, d\theta$$

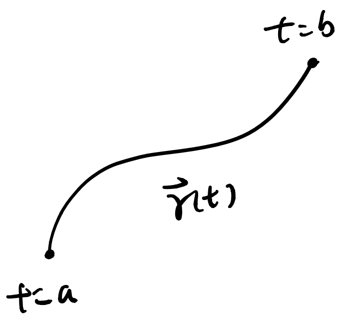
To generalize Area $= \displaystyle\iint_{D(u,v)} |\vec{r}_u \times \vec{r}_v| \cdot dA$

✓ Important.

Line Integral (of functions / of a vector field).

Question: I have a rope $\vec{\gamma}(t)$, $\rho(t)$ is the density function.

how heavy ?



$$\text{mass} = \int_C \rho(t)\cdot ds$$

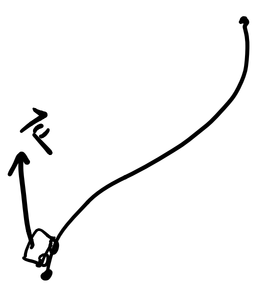
$$= \int_{t=a}^{t=b} \rho(t)\,\|\vec{\gamma}'(t)\|\cdot dt$$

← l.i of functions.

Question: How much work have you done?

l.i of vector field.

another way of writing.



$$\text{work} = \int_C \vec{F}(t)\cdot d\vec{\gamma}(t) = \int_C P(x,y,z)\cdot dx + Q(x,y,z)\,dy + R(x,y,z)\cdot dz$$

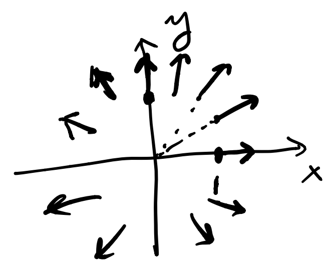
$$= \int_{t=a}^{t=b} \vec{F}(t)\cdot \vec{\gamma}'(t)\cdot dt \qquad = \int_C \begin{pmatrix} P(x(t),y(t),z(t)) \\ Q(\cdots\cdots) \\ R(\cdots\cdots) \end{pmatrix}\cdot \underbrace{\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}}_{= d\vec{\gamma}}$$

Vector Field is vector of functions.

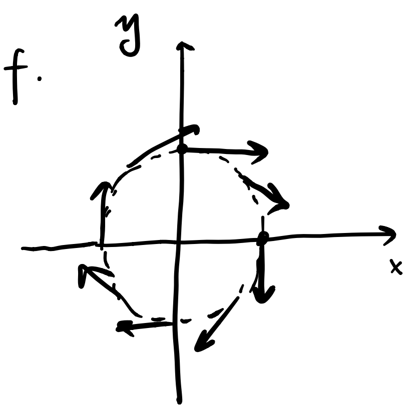
eg. vector field in x-y plane is simply

$$\vec{F}(x,y) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$$

eg. $\begin{pmatrix} x \\ y \end{pmatrix}$ is a vector field.



eg. $\begin{pmatrix} y \\ -x \end{pmatrix}$ is a v.f.



$$\vec{F} = \begin{pmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{pmatrix}$$

- ∇f is one common construction of vector field.
- **divergence** of vector field : $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = P_x + Q_y + R_z$

$$\vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

- **curl** of vector field : $\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{pmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{pmatrix}$

(*) • $\vec{\nabla} \times \nabla f = \vec{0}$

• $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$

Ex : $\vec{F} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad \vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix}$ $\quad 0 \le t \le 2\pi$.

compute $\int_C \vec{F} \cdot d\vec{r} = ?$

$$\int_0^{2\pi} \vec{F}(t) \cdot \underbrace{d\,\vec{r}(t)}_{= \vec{r}\,'(t)\,dt} = \int_0^{2\pi} \begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \, dt$$

$$= \int_0^{2\pi} 0 \cdot dt = 0$$