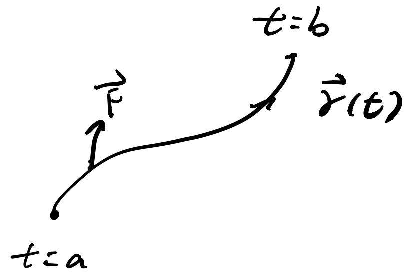
Recall line integral of vector fields

$$\int_C \vec{F}\, d\vec{r} = \int_{t=a}^{t=b} \vec{F}\cdot \vec{r}\,'(t)\, dt$$



Goal: introduce conservative vector fields.

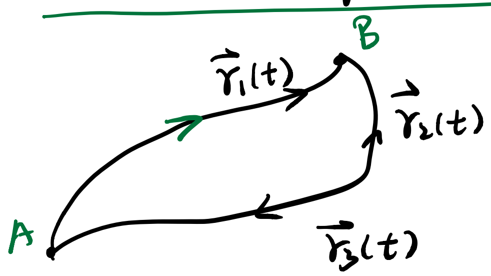
Def: $\vec{F}$ is conservative in a region R if $\vec{F} = \nabla f$.

by fundamental theorem of line integral. (recall before $\int_a^b f'(x)\,dx = f(b)-f(a)$)

$$\int_C (\nabla f)\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

this implies that $\int_C (\nabla f)\cdot d\vec{r}$ only depend on endpts

① does not depend on the path. ① <=> ②

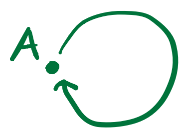


$$\int_{\vec{r}_2} \vec{F}\cdot d\vec{r} = -\int_{\vec{r}_3} \vec{F}\cdot d\vec{r}$$

equivalently. if we consider the loop, then ②

$$\int_{\vec{r}_1} \vec{F}\cdot d\vec{r} - \int_{\vec{r}_2} \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot d\vec{r} = 0$$

if C is a closed curve

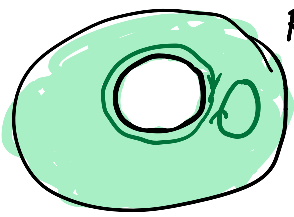


Q: How can we tell if a $\vec{F}$ is conservative? If $\vec{F}$ is conservative, how to find "anti-derivative" f?

Thm. Suppose $\vec{F} = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$ $\begin{matrix} P \\ Q \\ R \end{matrix}$ Both P and Q and their 1-st order partial derivatives are continuous. Given a region R that is a rectangle (R has no holes), then

$\vec{F}$ is conservative in R $\iff$ $P_y = Q_x$ in R.

$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \vec{0} \iff \begin{pmatrix} R_z - Q_y \\ P_z - R_x \\ Q_x - P_y \end{pmatrix} = \vec{0}$



R

eg. 1) $\vec{F} = \begin{pmatrix} yx^2 + x \\ \frac{x^3}{3} + y^2 \end{pmatrix}$

$R = \mathbb{R}^2$

Is this $\vec{F}$ conservative in R?

Yes.

$P_y = x^2 = Q_x$

2) What is a function f s.t. $\nabla f = \vec{F}$?

Method 1:

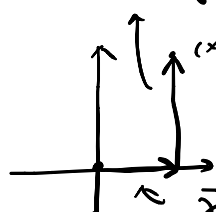
$$\int (yx^2 + x)\,dx = y \cdot \frac{x^3}{3} + \frac{x^2}{2} + C_1(y) \qquad C(y,z)$$

$$\int \left(\frac{x^3}{3} + y^2\right) dy = \frac{x^3}{3} \cdot y + \frac{y^3}{3} + C_2(x)$$

so take $C_1(y) = \frac{y^3}{3}$ $C_2(x) = \frac{x^2}{2}$. then

let $f = y \cdot \frac{x^3}{3} + \frac{x^2}{2} + \frac{y^3}{3}$.

$\vec{r}_2(t) = \begin{pmatrix} x \\ t \end{pmatrix}$ $0 \le t \le y$



(x,y)

$\vec{r}_1(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$ $0 \le t \le x$

Method 2: assume $f(0,0) = 0$ then

$$f(x,y) - f(0,0) = \int_{(0,0)}^{(x,y)} \vec{F} \cdot d\vec{r} = \left(\int_{\vec{r}_1} + \int_{\vec{r}_2} \right) \vec{F} \cdot d\vec{r}$$

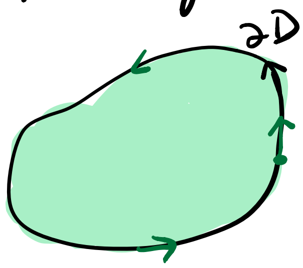
$$= \int_0^x \begin{pmatrix} t \\ \frac{t^3}{3} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt + \int_0^y \begin{pmatrix} tx^2 + x \\ \frac{x^3}{3} + t^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt$$

$$= \int_0^x t\,dt + \int_0^y \left(\frac{x^3}{3} + t^2\right) dt$$

$$= \frac{x^2}{2} + \frac{x^3}{3} y + \frac{y^3}{3}$$

Green's Thm.

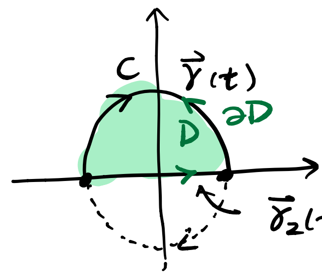
C: piecewise smooth closed curve that is the boundary of a region D.



$$\vec{F} = \begin{pmatrix} P \\ Q \end{pmatrix}$$

$$\oint_{\partial D} \vec{F} \cdot d\vec{\gamma} = \iint_D (Q_x - P_y)\, dA$$

eg. $\vec{F} = \begin{pmatrix} y \\ y^2 \sin(y+1) \end{pmatrix}$ Q: $\int \vec{F} \cdot d\vec{\gamma} = ?$



$\vec{\gamma}_2(t) = \begin{pmatrix} t \\ 0 \end{pmatrix} \quad -1 \le t \le 1$

$$\int_{-C} \vec{F} \cdot d\vec{\gamma} + \int_{\vec{\gamma}_2(t)} \vec{F} \cdot d\vec{\gamma} = \iint_D (-1)\, dA = -1 \cdot \pi^2.$$

$$\int_{\vec{\gamma}_2(t)} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot d\vec{\gamma} = 0$$