Surface Integral. $\vec{x}(u,v)$

- of functions

$$\iint_D f(u,v)\cdot dA = \iint_D f(u,v)\,\|\vec{x}_u \times \vec{x}_v\|\,du\,dv$$

if $f=1$

$$\iint_D 1\cdot\|\vec{x}_u \times \vec{x}_v\|\,du\,dv = \text{surface area}$$

(model: mass of surface, center of mass)

- of vector fields (Later Story ....)

II (flux)

unit normal

$$\iint_D \vec{v}\cdot\vec{N}\,dA = \iint_D \vec{v}\cdot\vec{N}\,\|\vec{x}_u\times\vec{x}_v\|\,du\,dv$$

$$\parallel$$

$$= \iint_D \vec{v}\cdot(\vec{x}_u\times\vec{x}_v)\,du\,dv$$

Line Integral $\vec{\gamma}(t)$

- of functions

$$\int f(t)\cdot ds = \int f(t)\,\|\vec{\gamma}'(t)\|\,dt$$

(model: compute mass of curve, center of mass)

if $f=1$

$$\int 1\cdot\|\vec{\gamma}'(t)\|\,dt = \text{arc length.}$$

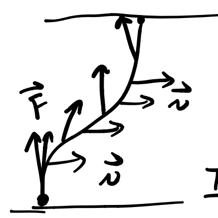
- of vector fields (Later Story: Green's Thm).

I (work)

$$\int \vec{F}\cdot\vec{\gamma}'(t)\,dt = \int \vec{F}\cdot\vec{T}\cdot\|\vec{\gamma}'(t)\|\,dt$$

unit vec

$$= \int \vec{F}\cdot\vec{T}\cdot ds$$



II (flux)

$$\int \vec{v}\cdot\vec{N}\cdot ds = \int \vec{v}\cdot\vec{N}\cdot\|\vec{\gamma}'(t)\|\,dt$$

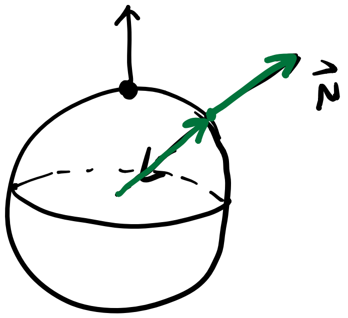
Example: sphere with radius R.

$$\vec{x}(\phi,\theta) = \begin{pmatrix} R\sin\phi\cos\theta \\ R\sin\phi\sin\theta \\ R\cos\phi \end{pmatrix} \qquad \begin{array}{l} 0\le\phi\le\pi \\ 0\le\theta<2\pi \end{array}$$

1) What is the area of the surface?

$$\vec{x}_\phi = \begin{pmatrix} R\cos\phi\cos\theta \\ R\cos\phi\sin\theta \\ -R\sin\phi \end{pmatrix} \qquad \vec{x}_\theta = \begin{pmatrix} -R\sin\phi\sin\theta \\ R\sin\phi\cos\theta \\ 0 \end{pmatrix}$$

normal vector $\quad \vec{X}_\phi \times \vec{X}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\phi \\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0 \end{vmatrix}$



$$= \begin{pmatrix} R^2\sin^2\phi\cos\theta \\ R^2\sin^2\phi\sin\theta \\ R^2\sin\phi\cos\phi(\cos^2\theta+\sin^2\theta) \end{pmatrix}$$

$$= R^2\sin\phi \cdot \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}$$

$$\int_0^{2\pi}\int_0^{\pi} \underbrace{\| \vec{X}_\phi \times \vec{X}_\theta \|}_{R^2\sin\phi} \cdot d\phi\, d\theta = R^2 \cdot 2 \cdot 2\pi = 4\pi R^2$$

2) If there is water flowing out of ~~the~~ the sphere with $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then what is the flux of water on this sphere.

Be careful to choose outward normal for $\vec{N}$ since the question asks for water flow "out" of the sphere.

$$\int_0^{2\pi}\int_0^{\pi} \vec{v} \cdot (\vec{X}_\phi \times \vec{X}_\theta)\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \underbrace{\begin{pmatrix} R\sin\phi\cos\theta \\ R\sin\phi\sin\theta \\ R\cos\phi \end{pmatrix} \cdot \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}} R^2\sin\phi \cdot d\phi\, d\theta$$

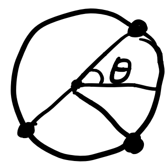
$$= \int_0^{2\pi} \int_0^{\pi} R^3 \sin\phi \cdot d\phi \, d\theta = R^3 \cdot 2 \cdot 2\pi = 4\pi R^3$$

3) If the mass is distributed as $\rho(\theta, \phi) = \cos^2\theta$.

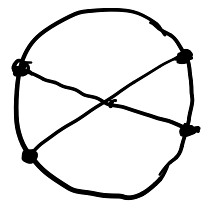
then what is the mass? what is the center of mass?

$$\int_0^{2\pi} \int_0^{\pi} \cos^2\theta \cdot \| \vec{x}_\phi \times \vec{x}_\theta \| \cdot d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \cos^2\theta \cdot R^2 \sin\phi \, d\phi \, d\theta = R^2 \cdot 2 \cdot \pi = 2\pi R^2$$



$\bar{z} = 0 \qquad \bar{x} = \bar{y} = 0$



Q: $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} + \dfrac{z^2}{b^2} = 1$

what is the area of this surface?

$$\vec{x}(\phi, \theta) = \begin{pmatrix} a \cdot \sin\phi \cos\theta \\ a \cdot \sin\phi \sin\theta \\ b \cdot \cos\phi \end{pmatrix}$$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$.

$\vec{x}_\phi = \cdots$

$\vec{x}_\theta = \cdots$