Review

Multivariable Calculus

Object:

$$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \qquad \vec{y} = f(x) = \begin{pmatrix} y_1 = f_1(x_1, \cdots, x_m) \\ \vdots \\ y_n = f_n(x_1, \cdots, x_m) \end{pmatrix}$$

Example: curve. $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$

$$t \longrightarrow \vec{\gamma}(t)$$

surface : $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$

vector field : $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\mathbb{R}^3 \longrightarrow \mathbb{R}^3$

Derivative

First order

- $f : \mathbb{R}^m \longrightarrow \mathbb{R}$. partial derivative.

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_m} \end{pmatrix}$$

Technique : Chain Rule.

∇f is normal to tangent plane.

linear approximation of f / tangent plane for $f = f(\vec{x}_0)$ at $\vec{x} = \vec{x}_0$.

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots \frac{\partial f}{\partial x_m} dx_m$$

$$\nabla_v f = \nabla f \cdot \vec{v} \quad \text{(directional derivative)}$$

Application : • Critical Points

• Conservative Vector Field

$\text{curl}(\vec{F}) = \vec{0} \Rightarrow \vec{F}$ is conservative (when D is nice)

• Velocity.

Second order.

- Taylor Expansion near a point.

$$f(x) \approx f(\vec{x}_0) + \vec{\nabla} f \cdot (\vec{x} - \vec{x}_0) + \underbrace{(\vec{x} - \vec{x}_0) \cdot H \cdot (\vec{x} - \vec{x}_0)^T}$$

when $n=2$. $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{xx} \end{pmatrix} \frac{1}{2}$.

Application: • determine local min/max.

• acceleration.

• curvature.

Integral

Fundamental Thm of Calculus.

- Double / Triple Integral.

strategy:

multiple layers after translating to iterated integral.

technique:

polar coordinate. (substitution) $r \cdot dr\, d\theta$

cylinder $r \cdot dr\, d\theta\, dz$.

sphere. $r^2 \sin\phi \, dr\, d\phi\, d\theta$.

application:

area

volume

arc length / surface area.

$\|\vec{r}'(t)\|$ $\|\vec{X}_u \times \vec{X}_v\|$

- Line Integral / Surface Integral.

L/S Integral of function: mass / center of mass

L/S Integral of vector fields: $\begin{cases} \text{flux} & \vec{F}\cdot\vec{n} \\ \\ \text{circulation.} & \vec{F}\cdot\vec{T} \end{cases}$

In reality $\vec{F}\cdot\vec{r}'(t)$, $\vec{F}\cdot(\vec{X}_u\times\vec{X}_v)$

Basic Strategy: Parametrization $\to$ Compute $\underbrace{\vec{F}\cdot\vec{n}}_{t,(u,v)}$ or $\underbrace{\vec{F}\cdot\vec{T}}_{t,(u,v)}$ as a fuction of L/S.

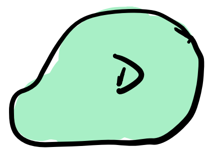
$\downarrow$

L/S integral of fuctions.

Technique: Green's Thm / Divergence Thm / Stoke's Thm.

Line Integral

Double Intergl



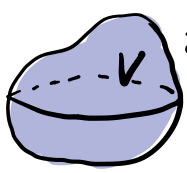
$$\oint_{\partial D} \vec{F}\cdot\vec{n}\,.ds = \iint_{D} \underbrace{(\nabla\cdot F)}_{\text{divergence of } \vec{F}.} dxdy.$$

$$\oint_{\partial D} \vec{F}\cdot\vec{T}\,.ds = \iint_{D} \text{curl}(\vec{F})\cdot\underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{=\vec{n}}.\,dxdy.$$

generalize.

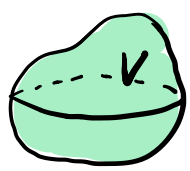
Divergence Thm.

Surface Integral



∂V: surface.

Triple Integral.



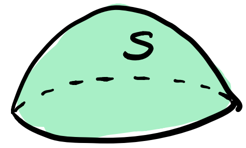
V: solid.

$$\oiint_{\partial V} \vec{F}\cdot\underbrace{\vec{n}}_{\text{outward}}\,dS = \iiint_{V} \vec{\nabla}\cdot\vec{F}\;dxdydz.$$

Stoke's Thm.

Line Integral

Surface Integral.



$$\oint_{\partial S} \vec{F}\cdot\vec{T}\cdot \underbrace{ds}_{=\,\|\vec{r}'(t)\|\,dt} \;=\; \iint_{S} (\mathrm{curl}\,\vec{F})\cdot\vec{n}\; \underbrace{dS}_{=\,\|\vec{x}_u\times\vec{x}_v\|\,du\,dv}$$