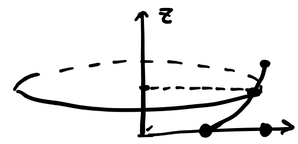
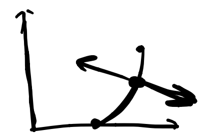
Math 212 Worksheet 10 Oct 29 2020, Instructor: Jiuya Wang

Problem 1 : Surface Integral Part 1: Area, Mass

Given a curve C defined by ~~$z = (x-1)^2$~~ $z = x^2$ in xz-plane with ~~$1 \le x \le 2$.~~ $2 \le x \le 3$ Rotate the curve by z-axis and get a surface R.



1. Give a parametrization of the surface R.
2. Compute the normal vector.
3. What is the area by surface integral?
4. Suppose the density function $\mu(x, y, z) = z$, compute the total mass of the surface?
5. What is the average z-coordinate?



1. $\vec{x}(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r^2 \end{pmatrix}$

2. $\vec{X}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 2r \end{pmatrix}$ $\quad \vec{X}_\theta = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix}$

$$\vec{X}_r \times \vec{X}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \begin{pmatrix} -2r\cdot r\cos\theta \\ -2r\cdot r\sin\theta \\ r \end{pmatrix}$$

3. $0 \le \theta < 2\pi$

$2 \le r \le 3$

$$\int_0^{2\pi}\int_2^3 r\cdot\sqrt{4r^2+1}\,dr\,d\theta$$

$$= 2\pi\cdot\int_2^3 \sqrt{4r^2+1}\,r\,dr$$

$u = 4r^2+1$

$$= 2\pi\cdot (4r^2+1)^{3/2}\cdot\frac{1}{12}\Big|_2^3$$

$$\int\sqrt{4r^2+1}\,r\,dr = \int \sqrt{u}\cdot\frac{du}{8} = \frac{1}{8}\cdot\frac{u^{3/2}}{3/2}$$

$u = 4r^2+1$, $du = 8r\,dr$

Problem 2 : Surface Integral Part 2: Flux

Consider the same sphere as in part 1 with outward normal,

1. Given the vector field $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, compute the flux?
2. Given the vector field $\vec{v} = \begin{pmatrix} y \\ -x \\ z \end{pmatrix}$, compute the flux?

4. $m = \int_0^{2\pi}\int_1^2 r^2\cdot r\cdot\sqrt{4r^2+1}\,dr$ $\quad u = 4r^2+1$ $\quad = \cdots$

5. $\bar{z} = \dfrac{\int_0^{2\pi}\int_1^2 \overset{z}{\underline{r^2}}\cdot\overset{\text{density}}{\underline{r^2}}\cdot r\cdot\sqrt{4r^2+1}\,dr}{m} = \cdots$

1. $$\int_0^{2\pi}\int_1^2 \begin{pmatrix} 2r\cdot r\cos\theta \\ 2r\cdot r\sin\theta \\ -r \end{pmatrix}\cdot\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r^2 \end{pmatrix} dr\,d\theta$$

pick $-\vec{X}_r\times\vec{X}_\theta$ to be outward.

$$= 2\pi\cdot\int_1^2 [2(r)\cdot r^2\cdot 2 - (r)^2\cdot r]\,dr = \cdots$$

2. $$\int_0^{2\pi}\int_1^2 \begin{pmatrix} \cdot \\ \cdot \\ -r \end{pmatrix}\begin{pmatrix} r\sin\theta \\ -r\cos\theta \\ r^2 \end{pmatrix} dr\,d\theta$$

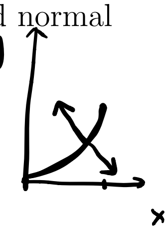
$$= 2\pi\int_1^2 -r(\quad)^2\,dr$$

$$= \cdots$$

Problem 3: Line Integral: Flux
Compute the following line integral:

1. $\vec{v} = \begin{pmatrix} x+y \\ 2y \end{pmatrix}$, $C : \vec{\gamma}(t) = (t, t^2), 0 \le t \le 1$, $\vec{N}$ the upward normal

2. $\vec{v} = \begin{pmatrix} xy^2 \\ x^2y \end{pmatrix}$, C: unit circle, $\vec{N}$ the outward normal



1. $\underline{\vec{\gamma}'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}}$

$\vec{N} = \begin{pmatrix} -2t \\ 1 \end{pmatrix}$

$$\int_0^1 \vec{v}\cdot\vec{N}\cdot dt = \int_0^1 \begin{pmatrix} t+t^2 \\ 2t^2 \end{pmatrix}\cdot\begin{pmatrix} -2t \\ 1 \end{pmatrix} dt$$

$$= \int_0^1 (\cancel{-2t^2} - 2t^3 + \cancel{2t^2})dt$$

$$= -t^3\Big|_0^1 = -1$$

2. $\vec{\gamma}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ $\quad 0 \le \theta < 2\pi \quad$ $\vec{\gamma}'(\theta) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

$\vec{N} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

$$\int_0^{2\pi} \begin{pmatrix} \cos\theta\cdot\sin^2\theta \\ \cos^2\theta\sin\theta \end{pmatrix}\cdot\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} d\theta$$

$$= \int_0^{2\pi} 2\cos^2\theta\sin^2\theta\, d\theta = \int_0^{2\pi} \frac{\sin^2 2\theta}{2}\, d\theta = \int_0^{2\pi} \frac{1}{4}\cdot(1-\cos 4\theta)\, d\theta$$

$$= \frac{1}{4}\cdot 2\pi$$

$$\int_0^{2\pi} \cos\theta\sin\theta\, d\theta = 0$$

If we use Green's Thm.

$$\oint \vec{v}\cdot\vec{N}\, dt = \iint_D (x^2+y^2)\, dxdy = \int_0^{2\pi}\int_0^1 r^2\cdot r\, drd\theta$$

$$= \left(\frac{r^4}{4}\right)\Big|_0^1 \cdot 2\pi$$

$$\int \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\cdot\begin{pmatrix} y_\theta \\ -x_\theta \end{pmatrix} d\theta = \int \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}\cdot\begin{pmatrix} x_\theta \\ y_\theta \end{pmatrix} d\theta$$

$$= \iint_D \Big[\overset{Q}{(v_1)_x} - \overset{P}{(-v_2)_y} \Big] \, dx\,dy$$

$$= \iint_D (v_1)_x + (v_2)_y \, dx\,dy$$