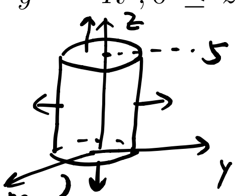
Problem: Divergence/Stokes Theorem

Compute the following flux surface integrals using divergence theorem / Stoke's theorem

1. S: the exterior of $T := \{(x,y,z) : x^2 + y^2 \le R^2, 0 \le z \le 5\}$ with outward normal $\vec{v} = \begin{pmatrix} x + e^{yz} \\ y + e^{xz} \\ z + e^{xy} \end{pmatrix}$, what is $\underbrace{\iint_S \cancel{\vec{F}} \cdot \vec{n}dS}_{v}$?

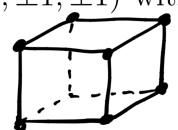


Q: Can you (Think about if S = [cylinder] with no cap/bottom) find $\vec{u}$ st $\text{curl}\,\vec{u} = \vec{v}$?!

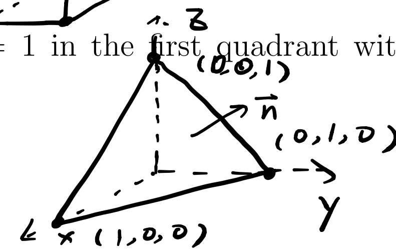
No!! $\text{div}(\vec{v}) = 3 \neq 0$!!

But $\text{div}(\text{curl}(*)) = 0$

2. S is the unit cube with vertexes $(\pm1, \pm1, \pm1)$ with outward normal. $\vec{v} = \begin{pmatrix} x^2 + e^{yz} \\ y^2 + e^{xz} \\ z^2 + e^{xy} \end{pmatrix}$, what is $\iint_S \vec{F} \cdot \vec{n}dS$?



3. S is the part of the plane $x + y + z = 1$ in the first quadrant with upward normal. $\vec{v} = \begin{pmatrix} ax \\ by \\ cz \end{pmatrix}$, what is $\iint_S \vec{F} \cdot \vec{n}dS$?

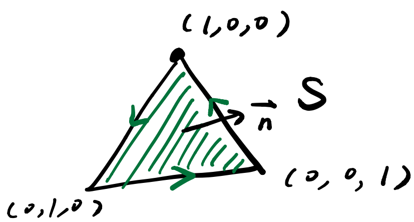


4. Given the vector field $\vec{F} = \begin{pmatrix} xy^2z^2 \\ yx^2z^2 \\ zx^2y^2 \end{pmatrix}$, γ is ~~the unit circle on xy-plane~~ any closed curve. What is $\int_\gamma \vec{F} \cdot d\vec{s}$?

5. γ is the triangle with vertex to be $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, and $\vec{F} = \begin{pmatrix} x + y^2 \\ y + z^2 \\ z + x^2 \end{pmatrix}$, what is $\int_\gamma \vec{F} \cdot d\vec{s}$? when γ is oriented conter-clock wise?

6. S is part of the surface $z = 4 - x^2 - y^2$ above z-axis with upward normal, and $\vec{F} = \begin{pmatrix} xe^{xy} \\ -ye^{xy} \\ -2 \end{pmatrix}$ what is $\iint_S \vec{F} \cdot \vec{n}dS$? $= \iint_S \vec{F} \cdot \vec{x}_u \times \vec{x}_v \, du dv$

5. By Stoke's Thm.



$$\text{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix} = \begin{pmatrix} -2z \\ -2x \\ -2y \end{pmatrix} = -2\begin{pmatrix} z \\ x \\ y \end{pmatrix}$$

$$\int_0^1 \int_0^{1-x} -2\begin{pmatrix} 1-x-y \\ x \\ y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} dy\, dx = \int_0^1 \int_0^{1-x} -2 \, dy\, dx$$

$$= -2 \cdot \int_0^1 (1-x)\, dx = \cdots$$

1. By divergence thm.

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_T (\underbrace{\vec{\nabla}\cdot\vec{F}}_{\text{divergence of } \vec{F}})\, dx\,dy\,dz$$

$$\vec{\nabla}\cdot\vec{F} = P_x + Q_y + R_z = 1 + 1 + 1 = 3$$

So $$\iiint_T \vec{\nabla}\cdot\vec{F}\, dx\,dy\,dz = 3\cdot \iiint_T 1\, dx\,dy\,dz$$

$$= 3\cdot Vol(T) = 3\cdot \pi R^2\cdot 5 = 15\pi R^2. \quad \square$$

2. By divergence thm.

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_T \vec{\nabla}\cdot\vec{F}\, dx\,dy\,dz.$$

$$\vec{\nabla}\cdot\vec{F} = 2x + 2y + 2z$$

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (2x+2y+2z)\, dx\,dy\,dz = 0$$

3. Method 1: parametrize.

$$\vec{x}(x,y) = \begin{pmatrix} x \\ y \\ 1-x-y \end{pmatrix} \qquad 0 \le x \le 1 \qquad 0 \le y \le 1-x$$

$$\int_0^1\int_0^{1-x} \begin{pmatrix} ax \\ by \\ c(1-x-y) \end{pmatrix}\cdot(\vec{x}_x \times \vec{x}_y)\, dx\,dy = \int_0^1\int_0^{1-x} [ax + by + c(1-x-y)]\, dx\,dy$$

$$= \dots$$

$$\vec{x}_x = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \vec{x}_y = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \vec{x}_x \times \vec{x}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$\vec{x}_x \times \vec{x}_y = \overbrace{\frac{(1,1,1)}{\sqrt{3}}}^{\vec{n}}\cdot \overbrace{\|\vec{x}_x \times \vec{x}_y\|}^{\sqrt{3}} = (1,1,1) \qquad = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

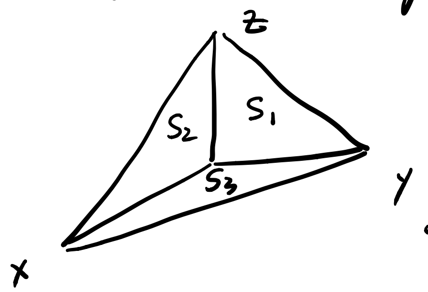
Rmk: if your plane is $ax+by+cz+d=0$.

$$\vec{X}(x,y) = \begin{pmatrix} x \\ y \\ (-d-ax-by)/c \end{pmatrix} \quad \text{if } c \neq 0.$$

$$\vec{X}_x = \begin{pmatrix} 1 \\ 0 \\ -a/c \end{pmatrix} \qquad \vec{X}_y = \begin{pmatrix} 0 \\ 1 \\ -b/c \end{pmatrix}$$

then $$\vec{X}_x \times \vec{X}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -a/c \\ 0 & 1 & -b/c \end{vmatrix} = \begin{pmatrix} a/c \\ b/c \\ 1 \end{pmatrix}$$

Method 2: use divergence thm.



4 faces give a closed surface.

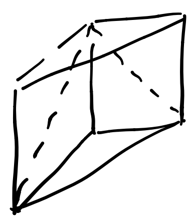
$$\iint_{S+S_1+S_2+S_3} \vec{F}\cdot\vec{n}\cdot dS = \iiint_T \vec{\nabla}\cdot\vec{F}\cdot dx\,dy\,dz$$

$$= (a+b+c)\cdot Vol(T)$$

$$= (a+b+c)\cdot \frac{1}{2}\cdot 1\cdot \frac{1}{3} = \frac{1}{6}(a+b+c)$$

Now for S_3: $(z=0)$

$$\vec{F} = \begin{pmatrix} ax \\ by \\ 0 \end{pmatrix} \qquad \vec{n} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$



$$\vec{F}\cdot\vec{n} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{so} \quad \iint_{S_3} \vec{F}\cdot\vec{n}\cdot dS = 0.$$

S_2: $(y=0)$

$$\vec{F} = \begin{pmatrix} ax \\ 0 \\ cz \end{pmatrix} \qquad \vec{n} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \quad \text{so} \quad \iint_{S_2} \vec{F}\cdot\vec{n}\cdot dS = 0.$$

Same for S_1. Then we know $\iint_S \vec{F}\cdot\vec{n}\cdot dS = \frac{1}{6}\cdot(a+b+c)$.

4. On xy-plane $z=0$. $\vec{F}=\vec{0}$. therefore $\int_\gamma \vec{F}\cdot d\vec{s}=0$.

$\text{Curl}\,\vec{F}=\vec{0}$. $\Rightarrow$ $\underbrace{\vec{F} \text{ is conservative}}_{\Updownarrow \ \text{curl}\,\vec{F}=0}$. so $\int_\gamma \vec{F}\cdot d\vec{s}=0$ for any closed path.

conditional on $\vec{F}$ defined everywhere smoothly.

6. Method 1. Notice that $\vec{F}'=\begin{pmatrix} y \\ -x \\ z+e^{xy}\end{pmatrix}$ has $\text{curl}\,\vec{F}'=\vec{F}$.

$$\text{curl}(\vec{F}')=\begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ y & -x & z+e^{xy}\end{vmatrix} = \begin{pmatrix} x\cdot e^{xy} \\ -ye^{xy} \\ -2\end{pmatrix},$$

By Stoke's Thm.

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \oint_\gamma \vec{F}'\cdot\vec{T}\,ds = \int_0^{2\pi} \begin{pmatrix} y(\theta) \\ -x(\theta) \\ z(\theta)+e^{xy}(\theta)\end{pmatrix}\cdot\begin{pmatrix} -y(\theta) \\ x(\theta) \\ 0\end{pmatrix} d\theta$$

$x^2+y^2=4$ on xy-plane.

$$= \int_0^{2\pi} -4 \, d\theta$$

$$= -4\cdot 2\pi = -8\pi.$$

$$\vec{\gamma}(\theta)=\begin{pmatrix} 2\cos\theta \\ 2\sin\theta\end{pmatrix}$$

$$\vec{\gamma}'(\theta)=\begin{pmatrix} -2\sin\theta \\ 2\cos\theta\end{pmatrix}$$

Method 2. Notice $\vec{\nabla}\cdot\vec{F}=0$ so by divergence thm.

$$\iint_{S+S_1} \vec{F}\cdot\vec{n}\cdot dS = \iiint_T 0\cdot dx\,dy\,dz = 0$$

where S_1 is the surface $x^2+y^2\le 4$ in xy-plane.

Therefore we have

$$\iint_S \vec{F}\cdot\vec{n}\,dS = -\iint_{S_1} \begin{pmatrix} x e^{xy} \\ -y e^{xy} \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dx\,dy$$

$$= -\iint_{S_1} 2 \cdot dx\,dy = -2 \cdot \text{Area}(S_1)$$

$$= -2 \cdot 4\pi = -8\pi$$