Problem 1 : Know the Curves
Describe the Shape of the Following Curve $\vec{x}(t)$:

1. $\begin{pmatrix} 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \end{pmatrix}$
2. $\begin{pmatrix} t \\ t^2 \end{pmatrix}$
3. $\begin{pmatrix} 3t \\ 4 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} t+1 \\ t-1 \\ 2t \end{pmatrix}$
4. $\begin{pmatrix} 3\cos \pi t \\ 3 \sin \pi t \end{pmatrix}$
5. $\begin{pmatrix} \cos \pi t \\ \sin \pi t \\ t \end{pmatrix}$
6. $\begin{pmatrix} 1 + 3\cos \pi t \\ -2 + 3 \sin \pi t \end{pmatrix}$

Problem 2: Tangent Vector and Tangent Line
Compute the tangent vector of following curves and write the parametrization of the tangent line at the given point.

1. ✓ $\begin{pmatrix} t^2 \\ t^3 \\ t^4 \end{pmatrix}$, $t = 1$

   $\vec{r}'(t) = \begin{pmatrix} 2t \\ 3t^2 \\ 4t^3 \end{pmatrix}$ $\quad \vec{r}(1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

   $\vec{r}'(1) = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ $\quad \vec{l}(s) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + s \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

   $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{4}$

2. $\begin{pmatrix} 2\theta - 2\sin\theta \\ 2 - 2\cos\theta \end{pmatrix}$, $\theta = \pi$

Problem 3: Computing Arc Length
Compute the arc length between given points.

1. $\begin{pmatrix}\cos\pi t\\ \sin\pi t\\ t\end{pmatrix}$ from $t=0$ to $t=\pi$

2. $\begin{pmatrix}e^t\cos t\\ e^t\sin t\end{pmatrix}$ from $t=0$ to $t=\pi$

3. $\begin{pmatrix}t-\sin t\\ 1-\cos t\end{pmatrix}$ from $t=0$ to $t=\pi$

$$\vec{\gamma}'(t) = \begin{pmatrix} e^t\cdot\cos t - e^t\cdot\sin t \\ e^t\cdot\sin t + e^t\cos t\end{pmatrix} = e^t\cdot\begin{pmatrix}\cos t-\sin t\\ \sin t+\cos t\end{pmatrix}$$

$$\|\vec{\gamma}'(t)\| = e^t\cdot\sqrt{(\cos t-\sin t)^2+(\sin t+\cos t)^2} = e^t\cdot\sqrt{2\cos^2 t+2\sin^2 t} = e^t\cdot\sqrt{2}$$

$$\int_0^\pi e^t\sqrt{2}\,dt = \sqrt{2}\cdot e^t\Big|_{t=0}^{t=\pi} = \sqrt{2}\,(e^\pi-1)$$

Problem 4: Curvature
Compute the curvature vector for the following curves, and compute the tangential and normal components of the acceleration vector.

1. $\begin{pmatrix}\cos\pi t\\ \sin\pi t\\ t\end{pmatrix}$

2. $\begin{pmatrix}e^t\cos t\\ e^t\sin t\end{pmatrix}$

$$\vec{M} = \frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt}\cdot\frac{dt}{ds}$$

$$\vec{T}(t) = \frac{\vec{\gamma}'(t)}{\|\vec{\gamma}'(t)\|} = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}\cos t-\sin t\\ \sin t+\cos t\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}-\sin t-\cos t\\ \cos t-\sin t\end{pmatrix}\cdot\frac{1}{\|\vec{\gamma}'(t)\|}$$

$$= \frac{1}{\sqrt{2}\cdot e^t\cdot\sqrt{2}}\cdot\begin{pmatrix}-\sin t-\cos t\\ \cos t-\sin t\end{pmatrix}$$

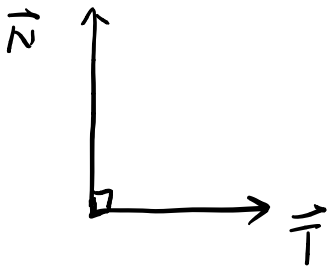
$$\varkappa = \|\vec{M}\| = \frac{1}{2\cdot e^t}\sqrt{2} = \frac{1}{\sqrt{2}\cdot e^t} \qquad \vec{N} = \frac{\vec{M}}{\varkappa}$$

Tangential / Normal Components of $\vec{a}$:

$$\vec{a} = \vec{r}''(t) = \frac{d(v(t)\cdot\vec{T}(t))}{dt} \quad (\text{because } \vec{r}'(t) = v(t)\cdot\vec{T}(t))$$

$$= \frac{d(v(t))}{dt}\cdot\vec{T}(t) + v(t)\cdot\frac{d\vec{T}(t)}{dt}$$

$$= \underbrace{v'(t)\cdot\vec{T}(t)}_{T\text{-con}} + \underbrace{v(t)^2\cdot \kappa\cdot\vec{N}}_{N\text{-con}}$$



$$\frac{d\,v(t)}{dt} = e^t\cdot\sqrt{2}$$

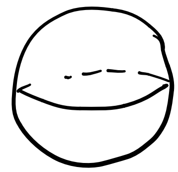
$$v^2\cdot\kappa = (e^t\cdot\sqrt{2})^2\cdot\frac{1}{e^t\cdot\sqrt{2}} = e^t\cdot\sqrt{2}$$

Problem 5 : Know the Surfaces

Describe the Shape of the Following Surface in 3-dimensional space.

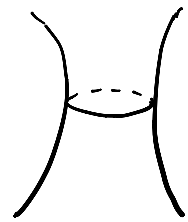
1. $x^2 + y^2 + z^2 = 1$
2. $x^2 + y^2 - z^2 = 1$
3. $z = x^2 + y^2$
4. $z^2 = x^2 + y^2$
5. $(x-1)^2 + y^2 + z^2 = 1$
6. $\frac{x^2}{4} + y^2 = 1$
7. $x^2 - y^2 = 1$

1.



5. $(1,0,0)$

2.



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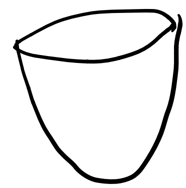

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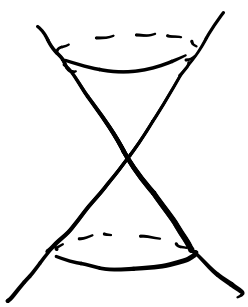

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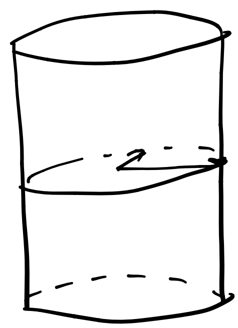

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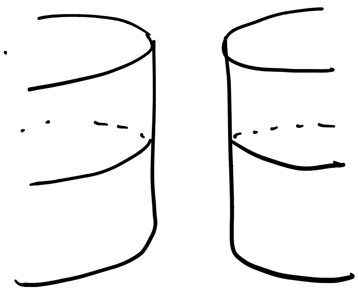


Problem 6 : Intersection Probelm

Describe the projection of the intersection of the given two surfaces.

Typo: 2

1. $y = 2x^{\cancel{3}\,2} + 3z^2$, $y = 5 - 3x^2 - z^2$, xz-plane
2. ✓ $y^2 = x^2 + z^2$, $x + y + z = 0$, xy-plane
3. $\cancel{x^2 + 4y^2 + 4(1 - x - y)^2 = 4}$, $x + y + z = 1$, xy-plane

Typo: $x^2 + 4y^2 + 4z^2 = 4$

2. (x, y, z) satisfies both equations.

$\uparrow \uparrow$

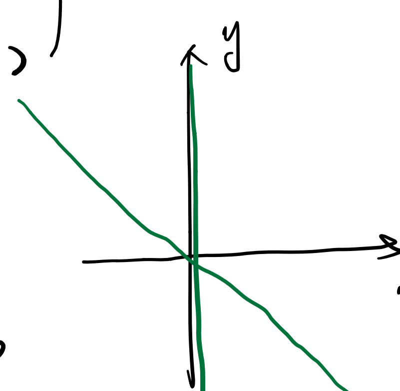
(x, y)

$$\begin{cases} y^2 = x^2 + z^2 \\ x + y + z = 0 \end{cases} \quad z = -(x+y)$$

$y^2 = x^2 + (x+y)^2 = 2x^2 + y^2 + 2xy$

$x \cdot (x+y) = 0$

$\Rightarrow x = 0$ or $x + y = 0$



$x=0$ $x+y=0$

1. $$\begin{cases} y = 2x^2 + 3z^2 \\ y = 5 - 3x^2 - z^2 \end{cases}$$

$$\Rightarrow \quad 2x^2 + 3z^2 = 5 - 3x^2 - z^2$$

$$5x^2 + 4z^2 = 5$$

$$x^2 + \frac{z^2}{5/4} = 1$$

3. $$\begin{cases} x^2 + 4y^2 + 4z^2 = 4 & ① \\ x + y + z = 1 & ② \end{cases}$$

$$\Rightarrow \quad z = 1 - x - y \qquad \text{plug in } ①.$$

$$x^2 + 4y^2 + 4\cdot(1 - x - y)^2 = 4$$

$$x^2 + 4y^2 + \cancel{4} + 4x^2 + 4y^2 - 8x - 8y + 8xy = \cancel{4}$$

$$5x^2 - 8x + 8y^2 - 8y + 8xy = 0$$