Problem 1 : Partial Derivatives/Tangent Plane

For following surfaces:

1) compute the partial derivatives;
2) write up the differential df;
3) find out the tangent plane at the given point;
4) what is the approximation at the second given point;
5) what is the normal vector of the plane you find out.

1. $z = xy^2$; $x = 2, y = 1, z = 2$; $x = 2.01, y = 0.09$
2. $z = \frac{xy}{x+y}$; $x = 3, y = 1, z = 3/4$; $x = 3.01, y = 1.02$
3. $z = e^{3-x^2-y^2}$; $x = 1, y = 1, z = e$; $x = 1.1, y = 1.1$

2. ① $\frac{\partial z}{\partial x} = \frac{y - xy}{(x+y)^2}$ $\quad \frac{\partial z}{\partial y} = \frac{x - xy}{(x+y)^2}$

② $df = \frac{y-xy}{(x+y)^2} dx + \frac{x-xy}{(x+y)^2} dy$

③ $df = \frac{-2}{16} dx + 0 \cdot dy$

$z - \frac{3}{4} = -\frac{1}{8} \cdot (x-3)$

④ $\Delta z \approx -\frac{1}{8} \cdot \Delta x = -\frac{1}{8} \cdot 0.01$

$z \approx \frac{3}{4} + \Delta z \approx \frac{3}{4} - \frac{1}{8} \cdot 0.01$

⑤ $\vec{N} = \begin{pmatrix} -\frac{1}{8} \\ 0 \\ -1 \end{pmatrix}$

Problem 2 : Gradient/Directional Derivatives

For above surfaces:

1) compute the gradient;
2) find the directional derivatives at given point along $\vec{u} = (1/2, \sqrt{3}/2)$;
3) find out the direction where the function increases/decreases the fastest.
4) find out the direction where the function remains the same.

① $\nabla f = \frac{1}{(x+y)^2} \begin{pmatrix} y - xy \\ x - xy \end{pmatrix}$

② $D_{\vec{u}} f = \nabla f \cdot \vec{u} = \begin{pmatrix} -\frac{1}{8} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} \\ \sqrt{3}/2 \end{pmatrix} = -\frac{1}{16}$

③ ∇f,

$-\nabla f$,

④ $\pm \begin{pmatrix} 0 \\ \frac{1}{8} \end{pmatrix}$

Problem 3: Chain Rule For the following functions:
1) Use chain rule to compute $\frac{df}{dt}$ (or $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$);
2) Evaluate $\frac{df}{dt}$ (or $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$) at given point;
3) Use chain rule to compute all second order derivatives in terms of t (or u and v);
4) Evaluate all second order derivatives in terms of t (or u and v) at given points.

1. ✓ $f(x,y) = x^2y^3 + x^3y^2$; $x(t) = t^2 + t$, $y(t) = e^t$; $t = 0$;
2. $f(x,y) = x^2 + y^2$; $x(t) = \cos t$, $y(t) = 2\sin t$; $t = \frac{\pi}{2}$;
3. $f(x,y,z) = xyz$; $x(t) = \ln t$, $y(t) = e^t$, $z(t) = \frac{1}{t}$; $t = 2$;
4. $f(x,y) = \sin x^2y + \cos xy^2$; $x(u,v) = e^{uv}$, $y(u,v) = \ln(uv)$; $u = 1, v = 1$;

$f = f(x(t), y(t))$

$$\begin{tikzcd} \overset{\mathbb{R}^1}{t} \arrow[rr] \arrow[dr] & & \overset{\mathbb{R}^1}{f} \\ & \underset{\substack{x(t) \\ y(t)}}{\mathbb{R}^2} \arrow[ur, "f(x,y)"'] & \end{tikzcd}$$

1.

① $$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= (y^3\cdot 2x + y^2\cdot 3x^2)\cdot(2t+1) + (x^2\cdot 3y^2 + x^3\cdot 2y)\cdot e^t = g(x,y,t).$$ ✓

$$\begin{tikzcd} & g \arrow[dl, dash] \arrow[d, dash] \arrow[ddl, dash] \\ x \arrow[d, dash] & y \arrow[dl, dash] \\ t & \end{tikzcd}$$

② $$\left.\frac{df}{dt}\right|_{t=0} = 0$$

③ $$\frac{d^2f}{dt^2} = \frac{\partial g}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial g}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial g}{\partial t}$$

$$= \left[(2t+1)\cdot(y^3\cdot 2 + y^2\cdot 3\cdot 2x) + e^t\cdot(2x\cdot 3y^2 + 3x^2\cdot 2y)\right]\cdot(2t+1)$$

$$+ \left[(2t+1)\cdot(3y^2\cdot 2x + 2y\cdot 3x^2) + e^t\cdot(x^2\cdot 6y + x^3\cdot 2)\right]\cdot e^t$$

$$+ (y^3\cdot 2x + y^2\cdot 3x^2)\cdot 2 + (x^2\cdot 3y^2 + x^3\cdot 2y)\cdot e^t$$

④ $$\left.\frac{d^2f}{dt^2}\right|_{t=0} = 1\cdot 2\cdot 1 = 2$$

$x = 0$
$y = 1$

$$t \xrightarrow{h} \vec{x} \xrightarrow{f} \vec{y} \xrightarrow{g} \vec{z}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \qquad \vec{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_\ell \end{pmatrix}$$

$\vec{x}(t)$ $\qquad$ $\vec{y}(t)$ $\qquad$ $\vec{z}(t)$

$$J_f = \left(\frac{\partial y_i}{\partial x_j} \right) \qquad J_g = \left(\frac{\partial z_i}{\partial y_j} \right)$$

$$\begin{array}{cc} & m \\ n & \begin{pmatrix} \quad \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \ell \\ m & \begin{pmatrix} \quad \end{pmatrix} \end{array}$$

$$J_h = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \end{pmatrix} = \vec{x}'(t)$$

$$J_f \cdot J_g = \begin{array}{cc} & m \\ n & \begin{pmatrix} \quad \end{pmatrix} \end{array} \begin{array}{cc} & \ell \\ m & \begin{pmatrix} \quad \end{pmatrix} \end{array} = \begin{array}{cc} & \ell \\ n & \begin{pmatrix} \quad \end{pmatrix} \end{array} = J_{f \circ g}$$

$\vec{x}(t)$: curve $\qquad \vec{y}(t) = f \circ h(t)$

$$\frac{d\vec{x}(t)}{dt} = \vec{x}'(t) = J_h$$

$$\vec{y}'(t) = J_{f \circ h} = J_f \cdot J_h = J_f \cdot \vec{x}'(t)$$

$$\vec{z}'(t) = J_{g \circ (f \circ h)} = J_g \cdot J_{f \circ h} = J_g \cdot J_f \cdot \vec{x}'(t)$$

A general formula for second order derivatives. (Only for fun, no need for general audience)

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = f_x(x,y)\cdot x'(t) + f_y(x,y)\cdot y'(t)$$

$$\frac{d^2f}{dt^2} = \left[\frac{d f_x(x,y)}{dt}\right]_{①} x'(t) + f_x(x,y)\cdot x''(t) +$$

$$\left[\frac{d f_y(x,y)}{dt}\right]_{②} y'(t) + f_y(x,y)\cdot y''(t)$$

$$= \left[f_{xx}(x,y)\cdot x'(t) + f_{xy}(x,y)\cdot y'(t)\right]_{①}\cdot x'(t) + f_x(x,y)\cdot x''(t)$$

$$\left[f_{yx}(x,y)\cdot x'(t) + f_{yy}(x,y)\cdot y'(t)\right]_{②}\cdot y'(t) + f_y(x,y)\cdot y''(t)$$

$$\begin{array}{ccccc} \mathbb{R}^1 & & \mathbb{R}^2 & & \mathbb{R}^1 \\ t & \xrightarrow{h} & \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} & \xrightarrow{s} & f(x,y) \end{array}$$

$$J_h = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \qquad J_s = \begin{pmatrix} f_x(x,y) & f_y(x,y) \end{pmatrix}$$

$$\therefore J_{s\circ h} = J_s\cdot J_h = \begin{pmatrix} f_x & f_y \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = f_x\cdot x' + f_y\cdot y'$$

$$\frac{\partial J_s(x,y)\cdot J_h(t)}{\partial t} = \left[\frac{\partial J_s}{\partial t}\right]\cdot J_h(t) + J_s\cdot\frac{\partial J_h}{\partial t}$$

$$= (J_w\cdot J_h)^T J_h(t) + J_s\cdot\frac{\partial J_h}{\partial t}.$$

$$\begin{array}{ccccc} \mathbb{R}^1 & & \mathbb{R}^2 & & \mathbb{R}^2 \\ t & \xrightarrow{h} & \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} & \xrightarrow{w} & \begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix} \end{array}$$

$$J_h = \begin{pmatrix} x' \\ y' \end{pmatrix} \qquad J_w = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

$(\ )^T$ means transpose of a matrix.

We use $(\ )^T$ here because the convention of J is $\begin{pmatrix} f_x'(t) \\ f_y'(t) \end{pmatrix}$, but we need $(f_x'(t)\ \ f_y'(t))$ to match.

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3. $z = e^{3-x^2-y^2}$; $x = 1, y = 1, z = e$; $x = 1.1, y = 1.1$

1. ① $\frac{\partial z}{\partial x} = y^2 \quad \frac{\partial z}{\partial y} = 2xy$

② $df = y^2 dx + 2xy\,dy$

③ $df\big|_{\vec{x_0}} = dx + 4dy$

$z - 2 = (x-2) + 4(y-1)$

④ $\Delta z \approx \Delta x + 4\Delta y$

$= 0.01 + 4\cdot(-0.01)$

$= -0.03$

$z \approx 2 - 0.03 = 1.97$

⑤ $\vec{N} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$

Problem 2 : Gradient/Directional Derivatives

For above surfaces:

1) compute the gradient;
2) find the directional derivatives at given point along $\vec{u} = (1/2, \sqrt{3}/2)$; unit vector.
3) find out the direction where the function increases/decreases the fastest.
4) find out the direction where the function remains the same.

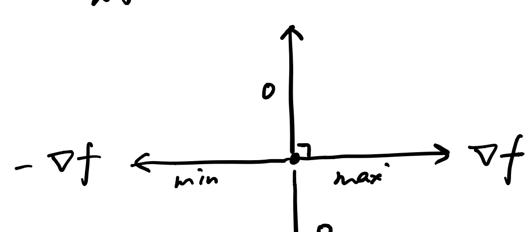
① $\nabla f = \begin{pmatrix} y^2 \\ 2xy \end{pmatrix}$

② $D_{\vec{u}} f\big|_{\vec{x_0}} = \begin{pmatrix} y^2 \\ 2xy \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}\Big|_{\vec{x_0}} = \frac{y^2}{2} + \sqrt{3}\,xy\Big|_{\vec{x_0}} = \frac{1}{2} + \sqrt{3}\cdot 2$

$\quad\quad = \nabla f \cdot \vec{u}$

③ ∇f , $-\nabla f$

④ $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ $\quad D_{\vec{u}} f = \nabla f \cdot \vec{u} = 0$ $\quad \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} -b \\ a \end{pmatrix}$



Problem 3: Chain Rule For the following functions:
1) Use chain rule to compute $\frac{df}{dt}$ (or $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$);
2) Evaluate $\frac{df}{dt}$ (or $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$) at given point;
3) Use chain rule to compute all second order derivatives in terms of t (or u and v);
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1. $f(x,y) = x^2y^3 + x^3y^2$; $x(t) = t^2 + t$, $y(t) = e^t$; $t = 0$;
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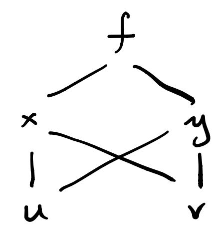
3. $f(x,y,z) = xyz$; $x(t) = \ln t$, $y(t) = e^t$, $z(t) = \frac{1}{t}$; $t = 2$; $x = e$ $y = 0$
4. $f(x,y) = \sin x^2y + \cos xy^2$; $x(u,v) = e^{uv}$, $y(u,v) = \ln(uv)$; $u = 1, v = 1$; f_{uu}

4. ① $\frac{\partial f}{\partial x} = 2xy\cdot\cos x^2y + y^2\cdot(-\sin xy^2)$ $\frac{\partial x}{\partial u} = v\cdot e^{uv}$ $\frac{\partial y}{\partial u} = \frac{1}{u}$

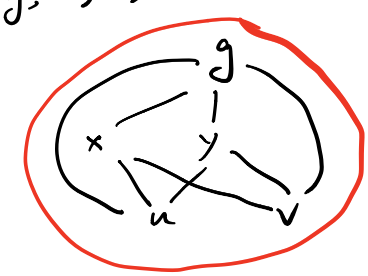
$\frac{\partial f}{\partial y} = x^2\cos x^2y + 2xy\cdot(-\sin xy^2)$ $\frac{\partial x}{\partial v} = u\cdot e^{uv}$ $\frac{\partial y}{\partial v} = \frac{1}{v}$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$



$$= (2xy\cdot\cos x^2y - y^2\cdot\sin xy^2)\cdot v\cdot e^{uv} + (x^2\cos x^2y - 2xy\cdot -\sin xy^2)\,\frac{1}{u}$$

$= g(x,y,u,v)$



$$\frac{\partial f}{\partial v} = (2xy\cdot\cos x^2y - y^2\cdot\sin xy^2)\cdot u\cdot e^{uv} + (x^2\cos x^2y - 2xy\cdot -\sin xy^2)\,\frac{1}{v}$$

② $\frac{\partial f}{\partial u}\Big|_{\vec{x}_0} = 1$ $\frac{\partial f}{\partial v}\Big|_{\vec{x}_0} = 1.$

③ $f_{uu} = \frac{\partial g}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial g}{\partial u}$ (Just one example)

$$= \Big(v\cdot e^{uv}\cdot[2y\cos x^2y + 2xy\cdot 2xy\cdot(-\sin x^2y) - y^2\cdot y^2\cdot\cos xy^2] + \frac{1}{u}\cdot[2x\cdot\cos x^2y + x^2\cdot 2xy\cdot -\sin x^2y - 2y\cdot(-\sin)xy^2 - 2xy\cdot y^2\cdot(-\cos xy^2)]\Big)\cdot u\cdot e^{uv}$$

$$+\Big(v\cdot e^{uv}\cdot\big[2x\cdot\cos x^2y+2xy\cdot(-\sin x^2y)\cdot x^2-2y\cdot\sin xy^2-y^2\cdot\cos xy^2\cdot x\cdot 2y\big]$$

$$+\frac{1}{u}\cdot\big[x^2\cdot(-\sin x^2y)\cdot x^2-2x\cdot -\sin xy^2-2xy\cdot(-\cos xy^2)\cdot 2xy\big]\Big)\cdot\frac{1}{u}$$

$$+\ (2xy\cdot\cos x^2y-y^2\cdot\sin xy^2)\cdot v^2e^{uv}+(x^2\cos x^2y-2xy\cdot -\sin xy^2)\frac{-1}{u^2}$$

④ $f_{uu}\big|_{\vec{x}_0}=1\cdot 2\cdot 1\cdot e+e\cdot 2-1=4e-1$

Problem 1 : Partial Derivatives/Tangent Plane
For following surfaces:
1) compute the partial derivatives;
2) write up the differential df; $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \cdots\cdots$
3) find out the tangent plane at the given point;
4) what is the approximation at the second given point;
5) what is the normal vector of the plane you find out.

1. $z = xy^2$; $x = 2, y = 1, z = 2$; $x = 2.01, y = 0.09$
2. $z = \frac{xy}{x+y}$; $x = 3, y = 1, z = 3/4$; $x = 3.01, y = 1.02$
3. $z = e^{3-x^2-y^2}$; $x = 1, y = 1, z = e$; $x = 1.1, y = 1.1$

3. ① $\begin{cases} \frac{\partial z}{\partial x} = e^{3-x^2-y^2} \cdot (-2x) \\ \frac{\partial z}{\partial y} = e^{3-x^2-y^2} \cdot (-2y) \end{cases}$

② $df = e^{3-x^2-y^2} \cdot (-2x)\, dx + e^{3-x^2-y^2} \cdot (-2y)\, dy$

③ $df = -2e\, dx - 2e\, dy$

$z - e = -2e(x-1) - 2e(y-1)$

⑤ $\vec{N} = \begin{pmatrix} -2e \\ -2e \\ -1 \end{pmatrix}$

④. $-2e \cdot 0.1 - 2e \cdot 0.1 \approx z - e \qquad z \approx 0.6e$

Problem 2 : Gradient/Directional Derivatives
For above surfaces:
1) compute the gradient;
2) find the directional derivatives at given point along $\vec{u} = (1/2, \sqrt{3}/2)$;
3) find out the direction where the function increases/decreases the fastest.
4) find out the direction where the function remains the same.

3. ① $\nabla f = e^{3-x^2-y^2} \cdot \begin{pmatrix} -2x \\ -2y \end{pmatrix}$

$\nabla f = \begin{pmatrix} a \\ b \end{pmatrix} \quad \begin{pmatrix} -b \\ a \end{pmatrix}$

② $D_{\vec{u}} f = \nabla f \cdot \vec{u} = e^{3-x^2-y^2} \cdot (-x - \sqrt{3} y)\big|_{x_0}$

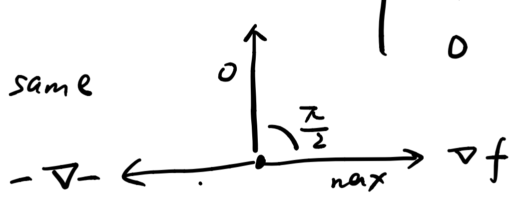
$= e \cdot (-1 - \sqrt{3})$

③ $\nabla f = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \cdot e$ increases

$-\nabla f = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \cdot e$ decreases

$\nabla f \cdot \vec{u} = \|\nabla f\| \cdot \cos\theta = \begin{cases} \text{maximal} & \cos\theta = 1 & \theta = 0 \\ \text{minimal} & \cos\theta = -1 & \pi \\ 0 & \cos\theta = 0 & \frac{\pi}{2} \end{cases}$

④ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ remains same



$0 \quad \frac{\pi}{2}$

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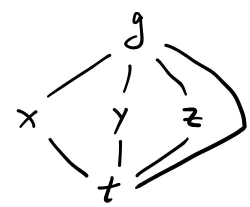
3. $f(x,y,z) = xyz$; $x(t) = \ln t$, $y(t) = e^t$, $z(t) = \frac{1}{t}$; $t = 2$;
4. $f(x,y) = \sin x^2y + \cos xy^2$; $x(u,v) = e^{uv}$, $y(u,v) = \ln(uv)$; $u = 1, v = 1$;

3. ① $\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$

$= yz\cdot\frac{1}{t} + xz\cdot e^t + xy\cdot\frac{-1}{t^2} = g(x,y,z,t)$

$x = x(t)$
$y = y(t)$
$z = z(t)$



② $\frac{df}{dt}\Big|_{t=2} = e^2\cdot\frac{1}{2}\cdot\frac{1}{2} + \ln 2\cdot\frac{1}{2}\cdot e^2 + \ln 2\cdot e^2\cdot\frac{-1}{4}$

$= \frac{e^2}{4}(1+\ln 2)$

③ $\frac{df'}{dt} = \frac{\partial g}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial g}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial g}{\partial z}\cdot\frac{dz}{dt} + \frac{\partial g}{\partial t}$

$= \left(z\cdot e^t + y\cdot\frac{-1}{t^2}\right)\cdot\frac{1}{t} + \left(z\cdot\frac{1}{t} + x\cdot\frac{-1}{t^2}\right)\cdot e^t + \left(y\cdot\frac{1}{t} + x\cdot e^t\right)\cdot\frac{-1}{t^2} + \left(yz\cdot\frac{-1}{t^2} + xz\cdot e^t + xy\cdot\frac{2}{t^3}\right)$

④ $\frac{df'}{dt}\Big|_{t=2} = \left(\frac{1}{2}\cdot e^2 + e^2\cdot\frac{-1}{4}\right)\cdot\frac{1}{2} + \left(\frac{1}{2}\cdot\frac{1}{2} + \ln 2\cdot\frac{-1}{4}\right)\cdot e^2 + \left(e^2\cdot\frac{1}{2} + \ln 2\cdot e^2\right)\cdot\frac{-1}{4} + \left(e^2\cdot\frac{1}{2}\cdot\frac{-1}{4} + \ln 2\cdot\frac{1}{2}\cdot e^2 + e^2\ln 2\cdot\frac{2}{2^3}\right)$

$= e^2\cdot\left(\frac{1}{8} + \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\right) + e^2\ln 2\cdot\left(-\frac{1}{4} - \frac{1}{4} + \frac{1}{2} + \frac{1}{4}\right)$

$= e^2\cdot\left(\frac{1}{8} + \frac{1}{4}\ln 2\right)$