Change of Variables (In preparation for $\left\{\begin{matrix}\text{polar} \\ \text{sphee.} \\ \text{cylinder}\end{matrix}\right\}$ coordinate).

$$\int_{x_0}^{x_1} \sin(x^2)\, dx \qquad x = u^2$$

$$\parallel \qquad \frac{dx}{du} = 2u.$$

$$\int_{u_0}^{u_1} \sin(u^2)\cdot 2u\, du$$

$$\Longrightarrow$$

$$\iiint_R f(x,y,z)\, dV = \iiint_W f(x(u,v,w), y(u,v,w), z(u,v,w))\, |\det(J_f)|\, d\sigma$$

If $\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$

$$f: \begin{pmatrix} u \\ v \\ w \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

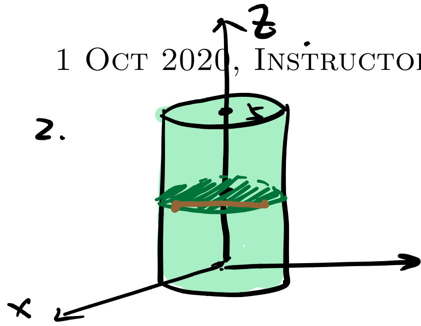
$$J_f = \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

MATH 212 WORKSHEET 6 1 OCT 2020, INSTRUCTOR: JIUYA WANG

Problem 1 : Describe Region

For all the region:
1. Sketch the region D;
2. Write the iterated integral on this region.

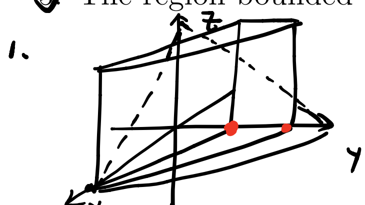
2.



$$\int_0^5\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx\,dz$$

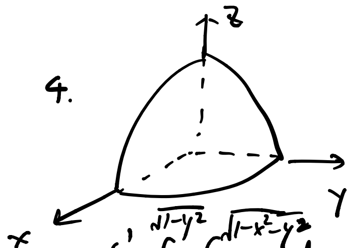
1. The region in the first octant bounded by $x+y+z=9$, $2x+3y=18$ and $x+3y=9$.
2. The region bounded by $x^2+y^2=1$ and $z=0$, $z=5$.
3. The region in the first octant bounded by $x^2+y^2=a^2$, and $z=x+y$.
4. The region in the first octant bounded by $x^2+y^2+z^2=1$.
5. The region bounded by $x^2+y^2+z^2=2$ and $z=\sqrt{x^2+y^2}$.

1.



$$\int_0^9\int_{\frac{9-x}{3}}^{\frac{18-2x}{3}}\int_0^{9-x-y} dz\,dy\,dx.$$

4.



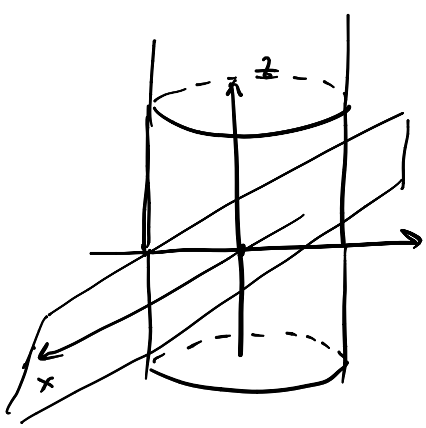
$$\int_0^1\int_0^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\,dx\,dy$$

Problem 2 : Change of Variable

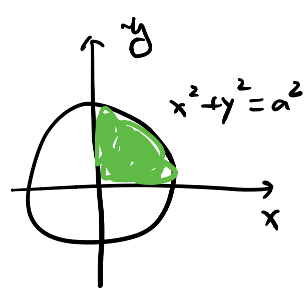
For the following change of variable, compute $\det(J)$:

1. $x=r\cos\theta, y=r\sin\theta$
2. $x=r\sin\phi\cos\theta, y=r\sin\phi\sin\theta, z=r\cos\phi$

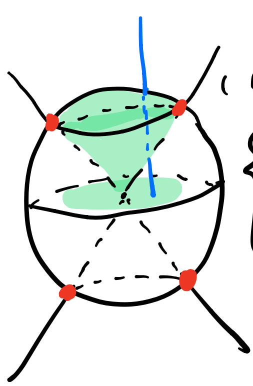
3.



$$\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{x+y} f(x,y,z)\,dz\,dy\,dx$$



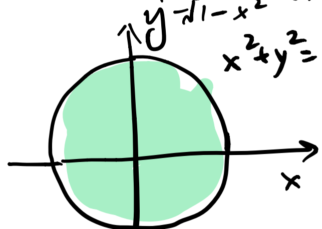
5.



(0, 2, 4)

$$\begin{cases} x^2+y^2+z^2=2 \\ z=\sqrt{x^2+y^2} \end{cases} \Rightarrow x^2+y^2=1$$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$



$$
\begin{array}{c}
\begin{array}{ccc} \qquad\qquad r & \qquad\qquad \phi & \qquad\qquad \theta \end{array} \\
J = \begin{pmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{pmatrix}
\end{array}
$$

$$
\begin{aligned}
\det J = {} & r^2 \cdot \left[\cos\phi\cos\theta \cdot \sin\phi\cos\theta \cdot \cos\phi + \sin^3\phi\sin^2\theta \right. \\
& \left. + \sin^3\phi\cos^2\theta + \sin\phi\sin^2\theta\cos^2\phi \right] \\
= {} & r^2 \left[\cos^2\phi\cos^2\theta\sin\phi + \sin^3\phi + \sin\phi\sin^2\theta\cos^2\phi \right] \\
= {} & r^2 \left[\cos^2\phi\sin\phi + \sin^3\phi \right] = r^2\sin\phi
\end{aligned}
$$

$$\bar{x} = \frac{\iiint_R x \cdot \mu \, dV}{\iiint_R \mu \cdot dV} \qquad \bar{y} = \frac{\iiint_R y \cdot \mu \cdot dV}{\iiint_R \mu \cdot dV}$$

Problem 3 : Application

$\bar{z} \ldots$

1. An object occupies the volume of the upper hemisphere $x^2 + y^2 + z^2 = 4$ and has density z at (x, y, z). Find the mass and the center of mass.
2. An object occupies the region inside the unit sphere at the origin, and has density equal to the square of the distance from the origin. Find the mass and the center of mass.

2.
$$m = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (x^2+y^2+z^2) \, dz\,dy\,dx$$

$$\bar{x} = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (x^2+y^2+z^2) \cdot x \, dz\,dy\,dx$$