Problem 1 : Compute Volume

Find the volume of the following regions:

$$\iiint_R 1\cdot dV = \iint_{D_R} h(x,y)\cdot dA.$$

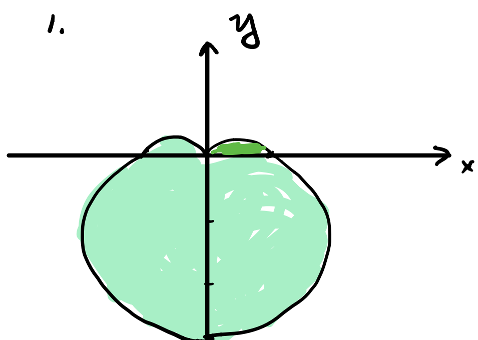
D_R projection of R onto xy-plane

1. The region bounded by $z = x^2 + y^2$ and $z = 4$;
2. The region (R) bounded by $x^2 + y^2 = 1$ and $z = 0$, $x + y + z = 0$; (top one)
3. The region in the first octant bounded by $x^2 + y^2 + z^2 = 1$.
4. The region bounded by $x^2 + y^2 + z^2 = 2$ and $z = 1$. (top one.)
5. The region bounded by $2x^2 + 2y^2 + z^2 = 3$ and $z = x^2 + y^2$.

Problem 2 : Compute the Area

1. The region bounded by $r = 1 - 2\sin\theta$.
2. The region bounded by $(x - 1)^2 + y^2 = 1$ in the first quadrant.

1.



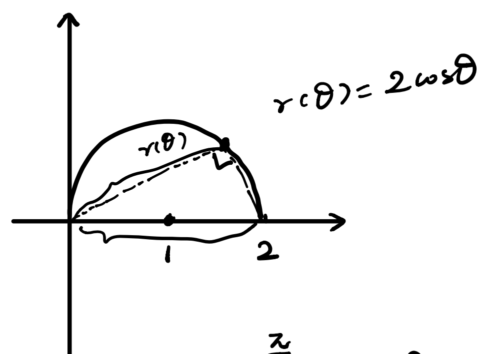
$1 - 2\sin\theta \geq 0 \Rightarrow \sin\theta \leq \frac{1}{2}$

$\Leftrightarrow \quad 0 \leq \theta \leq \frac{\pi}{6}$

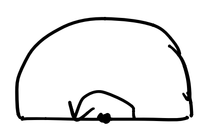
$\frac{5\pi}{6} \leq \theta \leq 2\pi.$

$$\int_0^{\frac{\pi}{6}} \int_0^{1-2\sin\theta} r\,.dr\,d\theta + \int_{\frac{5}{6}\pi}^{2\pi} \int_0^{1-2\sin\theta} r\,.dr\,d\theta$$

2.

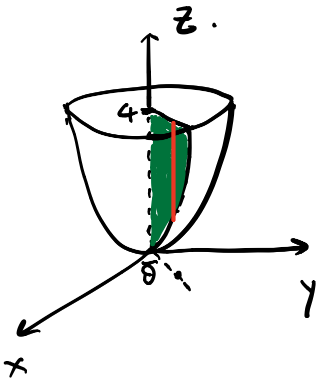


$$\iint_D 1\,dA = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r\,dr\,d\theta$$



$$\int_0^{\pi} \int_0^1 r\,dr\,d\theta$$

1.



projection to xy-plane is $x^2+y^2 \leq 4$.

$$\int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r\,dz\,dr\,d\theta.$$

$$\parallel$$

$$= \iint_D h(x,y)\,dA = \iint 4-(x^2+y^2)\,dA$$

$$D = \{x^2+y^2 \leq 4\}$$

$$\iiint_R \int_{z_0}^{z_1} 1\,dV$$

$$\parallel$$

$$\iint_D h(x,y)\,dA \qquad h(x,y) = z_1 - z_0$$

polar

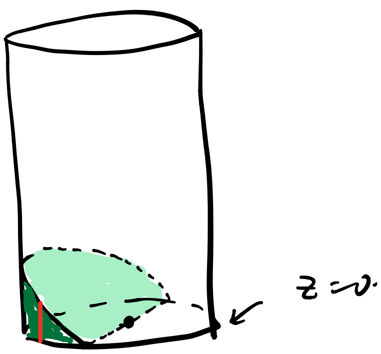
$$= \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 4-(x^2+y^2)\,dx\,dy.$$

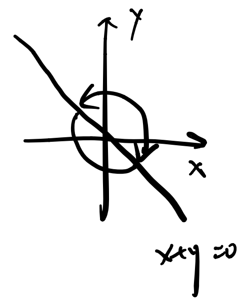
$$= \int_0^{2\pi}\int_0^{2} (4-r^2)\,r\,dr\,d\theta.$$

2.

$x+y \leq 0$

$$\int_{\frac{3\pi}{4}}^{\frac{7\pi}{4}}\int_0^{1}\int_0^{-r\cos\theta - r\sin\theta} r\,dz\,dr\,d\theta.$$





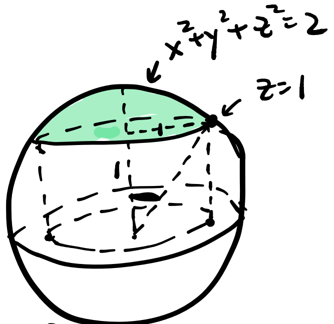
3.

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{1} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\frac{\sin\phi}{\cos^3\phi}\,d\phi = \frac{-d\cos\phi}{\cos^3\phi}$$

$$= d.\frac{\cos^{-2}\phi}{2}$$

4.



✓

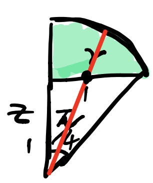
$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\frac{1}{\cos\phi}}^{\sqrt{2}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\parallel$$

$$\int_0^{2\pi}\int_0^{1}\int_1^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$$

$x^2+y^2=1$

$r=z=1$



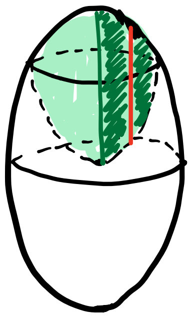
?. $\cos\varphi = 1$

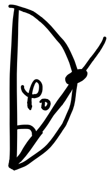
5. $2x^2+2y^2+z^2=3$ ← ellipsoid.

$z=x^2+y^2$ ← paraboloid.

Use cylinder coordinates:

$$\int_0^{2\pi}\int_0^{1}\int_{r^2}^{\sqrt{3-2r^2}} r\cdot dz\,dr\,d\theta.$$





$$\underbrace{2x^2+2y^2}_{=\,2r^2}+(\underbrace{x^2+y^2}_{t=r^2})^2=3$$

$$t^2+2t-3=0$$

$$(t-1)(t+3)=0$$

$$t=x^2+y^2=1$$

Problem 3 : Center of Mass
For the following region D, determine the center of mass.

1. Problem 1. 5, with density function $\mu(x, y, z) = (x^2 + y^2)^{1/2}$.
2. Problem 1. 3, with density function $\mu(x, y, z) = x$.
3. Compute the moment of inertia for Problem 1.5 with respect to z-axis.

1.

$$m = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{3-2r^2}} \mu \quad r \cdot dz\,dr\,d\theta.$$

$$= \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{3-2r^2}} r^2 \cdot dz\,dr\,d\theta.$$

$$= \int_0^{2\pi}\int_0^1 (r^2 \cdot \sqrt{3-2r^2} - r^4.)\,dr\,d\theta$$

$$= 2\pi \cdot \left[\int_0^1 (r^2 \cdot \sqrt{3-2r^2} \quad) \quad dr \quad - \frac{1}{5}\right]$$

$r = \sqrt{\frac{3}{2}}\sin\theta$

$dr = \sqrt{\frac{3}{2}}\cos\theta\,d\theta$

$$= 2\pi \cdot \int_0^{\frac{\pi}{2}} \left(\frac{3}{2} \cdot \sin^2\theta \cdot \sqrt{\frac{3}{2}}\cos\theta \qquad \sqrt{\frac{3}{2}} \cos\theta\,d\theta\right.$$

$$\int \sin^2\theta \cos^2\theta\,d\theta = \int \frac{1-\cos 2\theta}{2} \cdot \frac{1+\cos 2\theta}{2} \cdot d\theta$$

$$= \int \frac{1}{4} \cdot (1 - \cos^2 2\theta)\,d\theta$$

$$= \int \left(\frac{1}{4} - \frac{1}{4} \cdot \frac{1+\cos 4\theta}{2}\right).\,d\theta$$

$$\bar{x} = \frac{\iiint \mu r \cdot x \cdot dz\,dr\,d\theta}{m} = 0 \text{ by symmetry}$$

$\bar{y} = 0$ similarly

$$\bar{z} = \frac{\iiint \mu r z\,dz\,dr\,d\theta}{m}$$