MATH 212 WORKSHEET 7 OCT 15 2020, INSTRUCTOR: JIUYA WANG

Problem 1 : Line integral of Functions

Compute the following line integral:

$\vec{\gamma}' = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$

1. $f(x,y) = xy$, $C : \vec{\gamma}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}, 0 \le t \le 1$

2. $f(x,y,z) = x^2 + y^2$, $C : \vec{\gamma}(t) = \begin{pmatrix} 3\cos t \\ 3\sin t \\ t \end{pmatrix}, 0 \le t \le 2\pi$

3. If C is the quarter of the unit circle that lies in the first quadrant, then what is the average distance to the origin on C? What is the average polar angle θ?

$du = 8t\,dt$

$u = 1 + 4t^2$

1. $\int_0^1 f \cdot \|\vec{\gamma}'(t)\| \cdot dt = \int_1^5 \frac{u-1}{4} \cdot \sqrt{u}\, \frac{du}{8}$

$= \int_0^1 t \cdot t^2 \cdot \sqrt{1+4t^2}\, dt = \int_1^5 \frac{1}{32} \cdot u^{3/2} du$

$- \int_1^5 \frac{1}{32} u^{\frac{1}{2}} du$

2. $\int_0^{2\pi} (3^2\cos^2 t + 3^2 \sin^2 t) \cdot \sqrt{10}\, dt$

$= \int_0^{2\pi} 9 \cdot \sqrt{10} \cdot dt = 18 \cdot \sqrt{10}\,\pi$

3. $\vec{\gamma}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad 0 \le \theta \le \frac{\pi}{2}.$

$\int_0^{\frac{\pi}{2}} \cdot \|\vec{\gamma}'(\theta)\|\, d\theta = \frac{\pi}{2}$

$\int_0^{\pi/2} 1 \cdot \|\vec{\gamma}'(\theta)\|\, d\theta = \frac{\pi}{2}$

Ave distance $= 1$

$\int_0^{\frac{\pi}{2}} \theta \cdot \|\vec{\gamma}'(\theta)\|\, d\theta = \frac{\pi^2}{8}$

Ave angle $= \frac{\pi}{4}.$

Problem 2 : Line integral of Vector Fields

Compute the following line integral of vector fields:

1. $\vec{F} = \nabla f$ where $f(x,y) = x^2 + y^2 + xy$, $C : \vec{\gamma}(t) = (t, t^2), 0 \le t \le 1$

2. $\vec{F} = \text{curl} \begin{pmatrix} xy \\ yz \\ zx \end{pmatrix}$, $C : \vec{\gamma}(t) = \begin{pmatrix} 3\cos t \\ 3\sin t \\ t \end{pmatrix}, 0 \le t \le 2\pi$

1. $\nabla f = \begin{pmatrix} 2x + y \\ 2y + x \end{pmatrix}$

$\int_0^1 \vec{F} \cdot \vec{\gamma}'(t) \cdot dt$

$= \int_0^1 \begin{pmatrix} 2t + t^2 \\ 2t^2 + t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt$

$= \int_0^1 (2t + t^2 + 4t^3 + 2t^2)\, dt$

$= 3$

2. $\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \begin{pmatrix} -y \\ -z \\ -x \end{pmatrix}$

$\int_0^{2\pi} \begin{pmatrix} -3\sin t \\ -t \\ -3\cos t \end{pmatrix} \cdot \begin{pmatrix} -3\sin t \\ 3\cos t \\ 1 \end{pmatrix} dt$

$= \int_0^{2\pi} (9\sin^2 t - 3t\cos t - 3\cos t)\, dt$

$= 9\int_0^{2\pi} \sin^2 t\, dt - \int_0^{2\pi} 3t\cos t\, dt$

$= 9 \cdot \pi - 3 \cdot \left(- \int_0^{2\pi} \sin t\, dt \right) = 9\pi$

Problem 3 : Surface Area

$\sin(x)$

1. Find the area of the region R bounded by the curves $xy = 1$, $xy = 2$ and $\cancel{x}y^2 = 1$, $xy^2 = 2$.

2. Find the area of the surface that is the graph of $z = x + y^2$ for $0 \le x \le 1$ and $0 \le y \le 2$.

3. Find the area of the ellipse that is cut from the plane $2x + 3y + z = 6$ by the cylinder $x^2 + y^2 = 2$.

$$\Rightarrow \begin{cases} x = \frac{u^2}{v} \\ y = \frac{v}{u} \end{cases}$$

1. $$\begin{cases} u = xy \\ v = xy^2 \end{cases} \Rightarrow . \quad \det J = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ y^2 & 2xy \end{vmatrix} = 2xy^2 - xy^2 = xy^2$$

$$D := \{ (u,v) \mid 1 \le u \le 2,\ 1 \le v \le 2 \}$$

$$\iint_D 1 \cdot \frac{1}{xy^2} \, dA = \int_1^2 \int_1^2 \frac{1}{v} \, du \, dv$$

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= 1 \cdot (\ln 2 - \ln 1) = \ln 2.$$

2. $$\vec{r}(x,y) = \begin{pmatrix} x \\ y \\ x+y^2 \end{pmatrix} \quad \vec{r}_x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \vec{r}_y = \begin{pmatrix} 0 \\ 1 \\ 2y \end{pmatrix}$$

$$\| \vec{r}_x \times \vec{r}_y \| = \left\| \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 2y \end{vmatrix} \right\| = \left\| \begin{pmatrix} -1 \\ -2y \\ 1 \end{pmatrix} \right\| = \sqrt{4y^2 + 2}$$

$$\int_0^2 \int_0^1 \sqrt{4y^2+2} \, dx \, dy = \int_0^2 \sqrt{4y^2+2} \, dy$$

use $\tan^2\theta = \sec^2\theta - 1$

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3. $$\iint_{D:\, x^2+y^2 \le 2} \sqrt{1+4+9} \, dA = \sqrt{14} \cdot 2\pi \cdot \sqrt{2}.$$