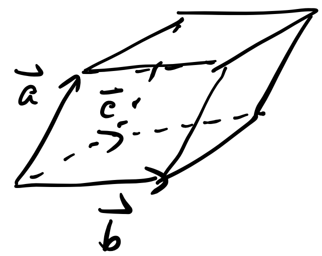
Week 7 Thursday.

Recall $\det(A)$ $A \in M_{3\times 3}(\mathbb{R})$

$\det(A) = \mathrm{Vol}(\langle \text{column vectors} \rangle)$



$\det(A)=0 \Leftrightarrow \mathrm{span}(\vec{a},\vec{b},\vec{c}) = \mathrm{span}(\text{two of them})$

$\Leftrightarrow$ $\vec{a},\vec{b},\vec{c}$ linear dependent

$\Leftrightarrow$ $A\cdot\vec{x}=0$ has non-trivial solutions

$\Leftrightarrow$ A is not injective $\Leftrightarrow$ not surjective

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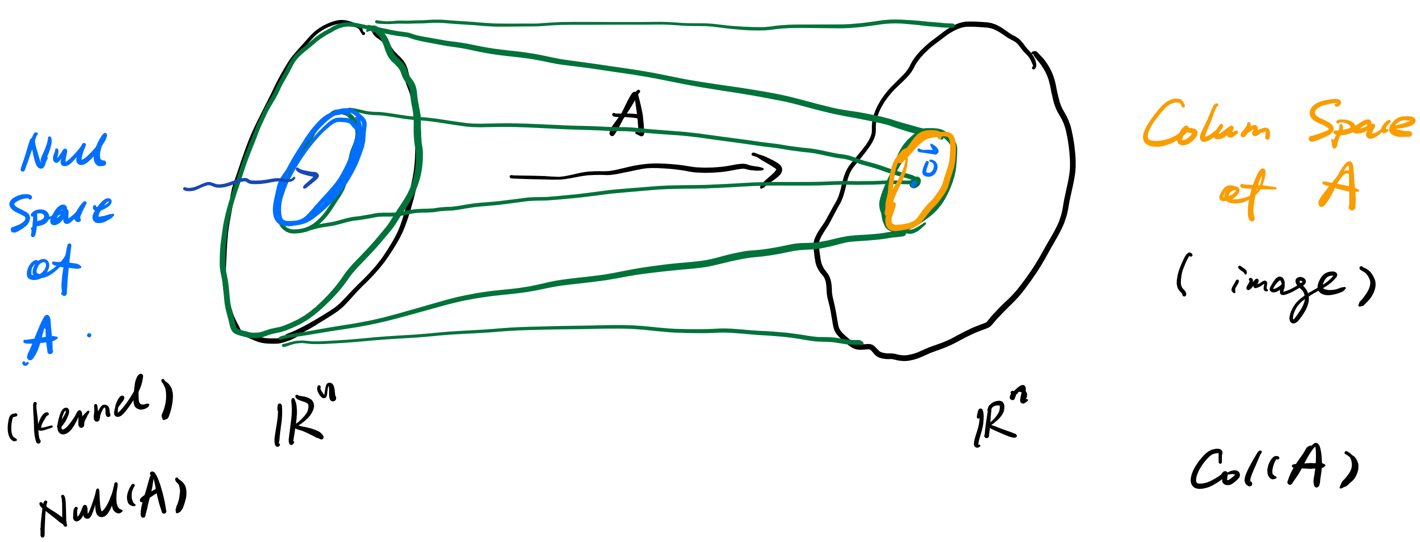
Q: How to describe A if A is not injective not surjective.

Ans: Rank.

Subspace:

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$



Subspaces of $\mathbb{R}^n$.

Def. A subset V of $\mathbb{R}^n$ is called a subspace if

① $\vec{0} \in V$;

② $\vec{u}, \vec{v} \in V$ then $\vec{u}+\vec{v} \in V$; (closed under addition)

③ $\vec{u} \in V$, then $\forall c \in \mathbb{R}$ $c \cdot \vec{u} \in V$

Prop. For any linear transformation A, the Null(A) and Col(A) are both subspaces of $\mathbb{R}^n$.

pf. Null(A) is clear a subspace using def of linear transformation.

Col(A): $\vec{0} = A(\vec{0})$ ✓

if $\vec{u} = A(\vec{a})$ $\vec{v} = A(\vec{b})$ then

$A(\vec{a}+\vec{b}) = A(\vec{a}) + A(\vec{b}) = \vec{u}+\vec{v}$ ✓

if $\vec{u} = A(\vec{a})$ then

$A(c \cdot \vec{a}) = c \cdot A(\vec{a}) = c \cdot \vec{u}$ ✓ □.

Def. (basis) If $V \in \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. and $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_r} \in V$ satisfy

① $\text{span} < \vec{v_1}, \vec{v_2}, \cdots, \vec{v_r} > = V$

② $\vec{v_1}, \cdots, \vec{v_r}$ linearly independ.

then we call $\{\vec{v_1}, \cdots \vec{v_r}\}$ is a basis for V.

(we also say V is a subspace ~~generated~~ spanned by $\{\vec{v_1}, \cdots, \vec{v_r}\}$

Rmk: basis is not unique for a given V.

if $\vec{v}_1, \vec{v}_2$ is a basis

then $\vec{v}_1+\vec{v}_2, \vec{v}_2$ is also a basis.

$$\lambda_1(\vec{v}_1+\vec{v}_2)+\lambda_2\vec{v}_2 = \lambda_1\vec{v}_1+(\lambda_1+\lambda_2)\vec{v}_2$$

Q: Is it possible to find two basis with different number of vectors?

Ans: No, it is not possible.

pf: Suppose $\{\vec{e}_1, \cdots, \vec{e}_r\}$ is one basis

$\{\vec{d}_1, \cdots, \vec{d}_t\}$ is another basis $r > t$.

then $\vec{e}_i$ is a linear combination of $\vec{d}_1, \cdots, \vec{d}_t$.

say $\vec{e}_i = \sum_{1\le j\le t} \alpha_{ij} \vec{d}_j$

Similarly $\vec{d}_j = \sum_{1\le i\le r} \beta_{ij} \vec{e}_i$

denote A to be the matrix with $A_{ij} = \alpha_{ij}$

B - - - - - $B_{ij} = \beta_{ij}$

then $A\cdot B = I_r$

$B\cdot A = I_t$

this is impossible if $r > t$.

since $B \in M_{t\times r}$ and echelon form of B has free variables.

$\exists \vec{x} \neq \vec{0}$ s.t $B\cdot\vec{x} = \vec{0}$ therefore $\vec{0} = A\cdot B\cdot\vec{x} = I_r\cdot\vec{x} = \vec{x}$

contradiction.

□.

Def (dimension) If $V \subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ and any basis of V contains r vectors, then we say r is the dimension of V.

eg. Null(A).

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 5 & 2 & 2 \end{pmatrix} \qquad A \cdot \vec{x} = \vec{0}$$

solve this system using row reduction

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 2 & 2 & 0 \end{array}\right) \xrightarrow[③ \to ③ - 5 \times ①]{② \to ② - 2①} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 2 & -8 & 0 \end{array}\right)$$

$$\xrightarrow{③ \to ③ - ② \times 2} \left(\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(↑ free variables.)

$$\vec{x} = \begin{pmatrix} -2x_3 \\ 4x_3 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$$

we get the basis $\left\{ \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} \right\}$ for Null(A).

Suppose the p.v.f is

$$x_3 \cdot \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} + x_5 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

① Null(A) is surely spanned by these vectors

② Since we get row echelon form. so the i-position in the parametric vector form is equal x_i if x_i is a free variable. therefore if $\vec{x} = \vec{0}$ then $x_i = 0$ for all free variable(s).