Week 8 Tuesday

Recall $Null(A)$ $Col(A)$

$\dim(W) =$ # of vectors in a basis of W.

To compute $Null(A)$: $A \xrightarrow{\text{row reduction}}$ echelon form. A'

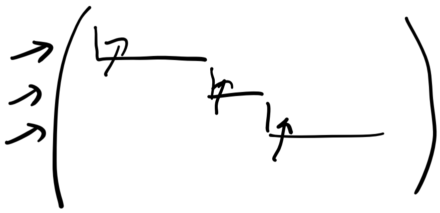
$\to$ solutions of $A'x=0$ in parametric vector form

$Col(A)$: subspace of $\mathbb{R}^m$ spanned by column vectors of A.

$Row(A)$: subspace of $\mathbb{R}^n$ spanned by row vectors of A.

Row operations just change the $Row(A)$.

Say row echelon form A'

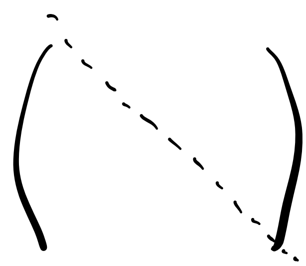


then the basis for $Row(A)$ is simply

Def Column operations $\begin{cases} \text{switch.} \\ \text{replace} \\ \text{scalar.} \end{cases}$

Def $A^T :=$ transpose of A.

$(A^T)_{ij} := A_{ji}$

$A =$ 

Def. (rank) $rk(A) := \dim(Col(A))$ for $A \in M_{m\times n}(\mathbb{R})$

Rmk: $rk(A)$ is also equal to # of pivot in echelon form of A.

$\dim(Null(A)) = \#$ of free variables

$= n - \#$ pivots

$= n - \dim(Row(A))$

Thm. If $A \in M_{n\times n}$, then

$$\dim(Null(A)) + \dim(Col(A)) = n.$$

Pf. Denote $\{\vec{e}_1, \cdots, \vec{e}_s\}$ to be a basis of $Null(A)$.

$\{\vec{d}_1, \cdots, \vec{d}_t\}$ to be a basis of $Col(A)$.

For each $\vec{d}_i$ we choose $\vec{v}_i$ s.t. $A(\vec{v}_i) = \vec{d}_i$.

Now we claim that $\{\vec{e}_1, \cdots, \vec{e}_s, \vec{v}_1, \cdots, \vec{v}_t\}$ form a basis of $\mathbb{R}^n$.

1). $\{\vec{e}_1, \cdots, \vec{v}_t\}$ spans $\mathbb{R}^n$:

$\forall\ \vec{w} \in \mathbb{R}^n$. $\quad A\cdot\vec{w} = \sum \beta_i \vec{d}_i$

$$\Rightarrow\ A\cdot(\vec{w} - \sum \beta_i \cdot \vec{v}_i) = A\vec{w} - \sum \beta_i \cdot A(\vec{v}_i)$$

$$= \vec{0}$$

$\Rightarrow\ \vec{w} - \sum \beta_i \vec{v}_i \in Null(A)$

So $\vec{w} - \sum \beta_i \vec{v}_i = \sum \alpha_i \vec{e}_i$ for some α_i

$\Rightarrow\ \vec{w} = \sum \alpha_i \vec{e}_i + \sum \beta_i \vec{v}_i$

2) Linear Indipendent:

Suppose $\vec{w} = \sum \alpha_i \vec{e}_i + \sum \beta_i v_i = \vec{0}$.

Then $A\cdot\vec{w} = \sum \beta_i \vec{d}_i = \vec{0} \Rightarrow \beta_i = 0$ since $\{\vec{d}_i\}$ are linearly indepadent.

and then $\sum \alpha_i \vec{e}_i = 0 \Rightarrow \alpha_i = 0$ since $\{\vec{\alpha}_i\}_i$ is linearly indepadent.

$\square$.

Deteminant

1.
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in M_{3\times3}(\mathbb{R})$$

$$a_{11}\cdot a_{22}\cdot a_{33} + a_{12}\cdot a_{23}\cdot a_{31} + a_{13}\cdot a_{21}\cdot a_{32}$$
$$\ominus \left(a_{13}\cdot a_{22}\cdot a_{31} + a_{12}\cdot a_{21}\cdot a_{33} + a_{11}\, a_{23}\, a_{32}\right)$$

2. Another equivalent definition is to use <u>minor</u>

$$a_{11}\cdot \underbrace{\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}}_{\text{minor}_{1,1}} - a_{12}\cdot \underbrace{\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}}_{\text{minor}_{1,2}} + a_{13}\cdot \underbrace{\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}}_{\text{minor}_{1,3}}$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\cdot d - bc$$

3. Or one can define $\det(A)$ to be

$$a_{11}\cdot m_{1,1} - a_{21}\cdot m_{21} + a_{31}\cdot m_{31}$$

$$\det(A^T) = \det(A)$$

Thm. $A \in M_{n\times n}(\mathbb{R})$ then.

1) switch. $\vec{r}_i$ and $\vec{r}_j$ $\quad R_{ij} \quad \det(R_{ij}\cdot A) = -\det(A)$

2) replace $\vec{r}_i$ by $\vec{r}_i + \lambda\vec{r}_j$ $\quad R_{i,j,\lambda} \quad \det(R_{i,j,\lambda} A) = \det(A)$

3) scalar $\vec{r}_i$ by λ $\quad R_{i,\lambda} \quad \det(R_{i,\lambda} A) = \lambda\cdot\det(A)$

Therefore, we can use echelon form to compute $\det(A)$.

$$\left|\begin{pmatrix} \vec{r}_1 = \vec{a} + \vec{b} \\ \vec{r}_2 \\ \vdots \\ \vec{r}_n \end{pmatrix}\right| = \left|\begin{pmatrix} \vec{a} \\ \vec{r}_2 \\ \vdots \\ \vec{r}_n \end{pmatrix}\right| + \left|\begin{pmatrix} \vec{b} \\ \vec{r}_2 \\ \vdots \\ \vec{r}_n \end{pmatrix}\right|$$

if. A' is an echelon form. then. we compute that.

$$\det(A') = (A')_{11} \times (A')_{22} \times \cdots \times (A')_{nn}$$

Use the 1st column det to compute determinant.

ex.

$$A = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$1 \leftrightarrow 2$

$$\begin{pmatrix} \boxed{1} & 1 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix} \qquad 4$$

$3 \rightarrow 3 - 1$

$4 \rightarrow 4 + 1$

$$\begin{pmatrix} \boxed{1} & 1 & 1 & -1 \\ 0 & \boxed{1} & -1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad -4$$

$3 \rightarrow 3 \times \frac{1}{2}$

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad -2$$

$4 \rightarrow 4 + 3$

$3 \rightarrow 3 + 2$

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad -2$$