Week 8 Thursday.

Recall determinant from last time

Compute det(A) via $\begin{cases} \text{formula} \\ \text{echelon form} \end{cases}$

$\det(A) \neq 0 \iff A$ is row equivalen to I

$\iff A$ is invertible.

$$\det(A) := \begin{cases} \sum_{\substack{i=i_0 \\ 1 \le j \le n}} a_{ij} \cdot m_{ij} \cdot (-1)^{i+j} \\ \sum_{\substack{j=j_0 \\ 1 \le i \le n}} a_{ij} \cdot m_{ij} \cdot (-1)^{i+j} \\ \sum_{\sigma} \left(\prod_{1 \le i \le n} a_{i\sigma(i)} \right) \cdot \operatorname{sgn}(\sigma) \end{cases}$$

where σ is bijection from $\{1, 2, \cdots, n\}$ to $\{1, 2, \cdots, n\}$
(inj and surj)

$\operatorname{sgn}(\sigma) := \# \{(i, j) \mid i < j, \sigma(i) > \sigma(j)\}$

eg. $\sigma: \begin{matrix} 1 \longrightarrow 1 \\ 2 \longrightarrow 3 \\ 3 \longrightarrow 2 \end{matrix}$ $\qquad \operatorname{sgn}(\sigma) = -1$

$a_{11} \cdot a_{23} \cdot a_{32} \cdot (-1)$

<u>Cramer's Rule.</u>

If $\det(A) \neq 0$, then A is invertible.

Moreover, we can give a formula of A^{-1} in terms of $\det(A)$, <u>minors.</u>

Define adj(A), adjugate matrix of A, as

$$[\operatorname{adj}(A)]_{ij} := m_{ji} \cdot (-1)^{i+j}.$$

We compute $A \cdot adj(A)$

$$[A \cdot adj(A)]_{ii} = \sum_{1 \le k \le n} a_{ik} \cdot [adj(A)]_{ki}$$

$$= \sum_{1 \le k \le n} a_{ik} \cdot m_{ik} \cdot (-1)^{i+k}$$

$$= det(A)$$

$$[A \cdot adj(A)]_{ij} = \sum_{1 \le k \le n} a_{ik} \cdot m_{jk} \cdot (-1)^{j+k}$$
$i \neq j$

$$= det \begin{pmatrix} a_{i1} & \cdots & a_{in} \\ a_{i1} & \cdots & a_{in} \end{pmatrix} \begin{matrix} \leftarrow i \text{ th row} \\ \leftarrow j \text{ th row.} \end{matrix}$$

$$= 0$$

$$A \cdot adj(A) = det(A) \cdot I \iff A \cdot \frac{adj(A)}{det(A)} = I$$

So $A^{-1} = \dfrac{adj(A)}{det(A)}$

eg. recall in mid-term.

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & k & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad det(A) = k + 0 + 0 - 0 - 6 - 4k$$
$$= -6 - 3k$$

$det(A) = 0 \iff A$ being non-invertible.

$$\iff 3k = -6 \iff k = -2.$$

$$adj(A) = \begin{pmatrix} k & -3 & -4k \\ -2 & -3 & 8 \\ -k & 3 & k-6 \end{pmatrix}$$

$$-3 = (-1)^{1+2} \cdot \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix}$$

$$-4k = (-1)^{3+1} \begin{vmatrix} 3 & 4 \\ k & 0 \end{vmatrix}$$

For $k=0$.

$$A^{-1} = \frac{adj(A)}{\det(A)} = \frac{1}{-6} \cdot \begin{pmatrix} 0 & -3 & 0 \\ -2 & -3 & 8 \\ 0 & 3 & -6 \end{pmatrix}$$

eg. Detemine the value of s. s.t.

$$\begin{cases} s\underline{x_1} - 2\underline{x_2} = 1 \\ 4s\underline{x_1} + 4s\underline{x_2} = 2 \end{cases}$$

has a unique solution.

$$A = \begin{pmatrix} s & -2 \\ 4s & 4s \end{pmatrix}$$

linear system $A \cdot \vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

A is row equivalent to $I \iff A$ is invertible

$\Updownarrow$ $\qquad\qquad \iff \det(A) \neq 0$

$\exists!$ solution for $A\vec{x} = \vec{b}$ for evey $\vec{b} \in \mathbb{R}^n$.

$$\det(A) = 4s^2 + 8s = 4s \cdot (s+2) \neq 0 \iff s \neq 0, -2.$$

Important Property of det:

$\det(AB) = \det(A) \cdot \det(B)$ for square matrix

pf. Recall that. row operations can be described by matrix multiplications.

1) swap i j rows then.

$$\underbrace{\begin{array}{cc} & \begin{array}{ccccc} & i & & j & \end{array} \\ \begin{array}{c} \\ i \to \\ \\ j \to \\ \\ \end{array} & \begin{pmatrix} 1 & & & & \\ & 0 & & 1 & \\ & & \ddots & & \\ & 1 & & 0 & \\ & & & & 1 \end{pmatrix} \end{array}}_{R_{ij}} \cdot A$$

so $\det(R_{ij} \cdot A) = \det(R_{ij}) \cdot \det(A)$

$\det(R_{ij}) = -1$

2) replace row i with row i + λ row j.

$$R_{ij\lambda} = \begin{array}{cc} & \begin{array}{ccccc} & & i & & j \end{array} \\ \begin{array}{c} \\ i \to \\ \\ j \to \\ \end{array} & \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & \lambda & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \end{array}$$

$\det(R_{ij\lambda}) = 1$ since it is upper / lower triangular

$\Rightarrow \det(R_{ij\lambda} A) = \det(R_{ij\lambda}) \cdot \det(A)$

3) scalar multiplication for row i

$$R_{i\lambda} = \begin{array}{cc} & \\ \begin{array}{c} \\ \\ i \to \\ \\ \end{array} & \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \lambda & \\ & & & \\ & & & 1 \end{pmatrix} \end{array}$$

$\det(R_{i\lambda}) = \lambda$.

$\Rightarrow \det(R_{i\lambda} A) = \det(R_{i\lambda}) . \det(A)$.

reduced echelon form

Now $\det(AB) = \det(\underbrace{R_1 R_2 \cdots R_e \cdot A_0}_{A} \cdot B)$

$= \det(R_1) . \det(R_2 \cdots R_e \cdot A_0 \cdot B)$

$\vdots$

$= \det(R_1) \cdots \det(R_e) . \det(A_0 \cdot B)$

if A is invertible. then $A_0 = I$.

so $\det(AB) = \det(R_1) \cdots \det(R_e) \cdot \det(B)$

$= \det(A) . \det(B)$.

if A is not-invertible then $A_0 \cdot B$ is not invertible so $\det(AB) = \det(A) \cdot \det(B) = 0$ $\square$.

Vector Space

the most common examples are $\mathbb{R}^n$

A vector space V over $\mathbb{R}$ is a set with $(+, \cdot)$ s.t.

1. $\forall u, v \in V$, $u+v \in V$, $c \cdot u \in V$;
2. $(u+v)+w = u+(v+w)$
3. $u+v = v+u$
4. $c(u+v) = cu+cv$ $\quad \forall c, d \in \mathbb{R}$

   $(c+d)u = cu+du$
5. $\exists\, 0$ s.t. $0+u = u$
6. $1 \cdot u = u$
7. $\forall u \in V$ $\exists!\ (-u)$ s.t. $u+(-u) = 0$
8. $c \cdot (d u) = (cd) \cdot u$ $\quad \forall c, d \in \mathbb{R}$ $\quad u \in V$

Ex: $V = M_{m \times n}(\mathbb{R})$. (matrix addition, scalar multiplication)

same with $\mathbb{R}^{m \times n}$