Week 9 Tuesday

Recall last time

Def. vector space (over $\mathbb{R}$) $(V, +, \cdot)$

1. $+, \cdot$ are closed in V.

   $+: V \times V \longrightarrow V$

   $\cdot: \mathbb{R} \times V \longrightarrow V$

2. $\forall \vec{u}, \vec{v} \in V$, $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

3. $\exists! \ \vec{0} \in V$ s.t. $\forall \vec{u} \in V$ $\vec{u} + \vec{0} = \vec{u}$

4. $\forall \vec{u} \in V \ \exists! \ \vec{v}$ s.t $\vec{u} + \vec{v} = \vec{0}$ (denote this $\vec{v}$ by $-\vec{u}$)

5. $\forall \vec{u}, \vec{v}, \vec{w} \in V$ $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

6. $\forall \vec{u} \ \vec{v} \in V$ $\forall c \in \mathbb{R}$. $c \cdot (\vec{u} + \vec{v}) = c \cdot \vec{u} + c \cdot \vec{v}$

   $\forall c, d \in \mathbb{R}$ $(c + d) \cdot \vec{u} = c \cdot \vec{u} + d \cdot \vec{u}$

7. $\forall \vec{u} \in V$ $\forall c, d \in \mathbb{R}$

   $(c \cdot d) \cdot \vec{u} = c \cdot (d \cdot \vec{u})$

8. $\forall \vec{u} \in V$ $1 \cdot \vec{u} = \vec{u}$.

Examples:

⓪ $V = \mathbb{R}^n$ $+$: vector addition

$\cdot$: scalar multipliation for vectors.

① $V = M_{m\times n}(\mathbb{R})$ $+$: matrix addition

$\cdot$: scalar multiplication for matrix

② $V = \mathbb{C} := \{ a + bi \mid a, b \in \mathbb{R} \}$ i is $\sqrt{-1}$.

$\oplus$: $(a_1 + b_1 i) \oplus (a_2 + b_2 i) = a_1 + a_2 + (b_1 + b_2) \cdot i$

$\odot$: $c \odot (a + bi) = c \cdot a + c \cdot bi$

V is "the same" with $\mathbb{R}^2$

$\mathbb{C} \xleftrightarrow{\quad f \quad} \mathbb{R}^2$ $\qquad$ f respects $+$ and $\cdot$.

$a+bi \qquad \begin{pmatrix} a \\ b \end{pmatrix}$

① $f(a+b) = f(a) + f(b)$

② $c \cdot f(a) = f(c \cdot a)$

$a_1+b_1 i \oplus a_2+b_2 i \qquad \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 \\ b_1+b_2 \end{pmatrix}$

$\quad \parallel$

$a_1+a_2 + (b_1+b_2)i$

$\mathbb{C}$ is isomorphic to $\mathbb{R}^2$ as a v.s. (over $\mathbb{R}$).

Def. Given V, W two v.s. $/\mathbb{R}$. we say V is isomophic to W if $\exists$ a bijection $f: V \to W$ that preserves $+$ and $\cdot$.

Fix n.

③ $V = \{ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid a_i \in \mathbb{R} \}$.

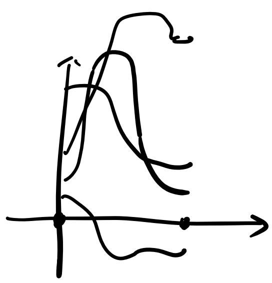
$+, \cdot$ usuall operation

④ $V = \{ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid a_i \in \mathbb{R},\ n \in \mathbb{Z}_{\geq 0} \}$

$+, \cdot$ usually operation.

still a vector space.

⑤. $V = \{ f: [0,1] \to \mathbb{R}$ continuous $\}$ $\qquad \lim_{x\to a} f(x) = f(a)$



$f \oplus g : \begin{matrix} [0,1] \longrightarrow \mathbb{R} \\ x \longrightarrow f(x)+g(x) \end{matrix}$

$\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

Sum of continuous fuctions are still continuous.

$c \odot f: \ x \longrightarrow c \cdot f(x)$

Fix W a v.s over $\mathbb{R}$.

⑥ $V = \{$ linear transformations $T: W \to \mathbb{R}^n \}$

$\oplus$: $T_1, T_2 : W \longrightarrow \mathbb{R}^n$

$T_1 \oplus T_2 : W \longrightarrow \mathbb{R}^n$

$\vec{w} \longrightarrow T_1(\vec{w}) + T_2(\vec{w})$

$\odot$: $c \odot T : \vec{w} \longrightarrow c \cdot T(\vec{w})$

Check that $P = T_1 \oplus T_2$ is still linear transformation:

$$P(\vec{w}_1 + \vec{w}_2) = T_1(\vec{w}_1 + \vec{w}_2) + T_2(\vec{w}_1 + \vec{w}_2) = T_1(\vec{w}_1) + T_1(\vec{w}_2) + T_2(\vec{w}_1) + T_2(\vec{w}_2)$$

$$\stackrel{\vee}{=} P(\vec{w}_1) + P(\vec{w}_2)$$

similarly for $c P(\vec{w}_1) = P(c \vec{w}_1)$

⑦ $V = \{$upper/lower Triangular matrix $\in M_{n \times n}(\mathbb{R})\}$

$V = \{$ diagonal matrix $\in M_{n \times n}(\mathbb{R})\}$

$+, \cdot$ matrix operation.

Subspace:

Def. V is v.s $/\mathbb{R}$. $H \subseteq V$ is a subset s.t.

① $\vec{0} \in H$

② H is closed under addition and scalar multiplication.

Rmk. H is just a subset and $(H, +, \cdot)$ forms a v.s.

eg. Def (span). $S = \{\vec{v}_1, \cdots, \vec{v}_n\}$ $\vec{v}_i \in V$.

$$\text{span}(S) := \{ \alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n \mid \alpha_i \in \mathbb{R} \}$$

is a subspace of V.

eg. Def (linear transformation) $T: V \longrightarrow W$

① $T(u+v) = T(u) + T(v)$

② $T(cu) = c \cdot T(u)$

then T is linear transformation.

Kernel of $T := \{ \vec{v} \in V \mid T(\vec{v}) = \vec{0} \} \subseteq V$

(Null(T)) if $T(\vec{v}_1) = T(\vec{v}_2) = \vec{0}$ then

$$T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{0}$$

$$\Rightarrow \vec{v}_1 + \vec{v}_2 \in Null(T).$$

Range of $T := \{ T(\vec{v}) \in W \mid \vec{v} \in V \} \subseteq W$

(Col(T)) if $T(\vec{v}_1) = \vec{w}_1$ $T(\vec{v}_2) = \vec{w}_2$

$$T(\vec{v}_1 + \vec{v}_2) = \vec{w}_1 + \vec{w}_2 \Rightarrow \vec{w}_1 + \vec{w}_2 \in Col(T).$$

if $T(\vec{v}) = \vec{w}$ then $T(\vec{v} + \vec{a}) = \vec{w}$ $\forall \vec{a} \in Kernel(T)$.

$\left\{ \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ 0 & & & 1 \end{pmatrix} \middle| \; * \in \mathbb{R} \right\}$ ✗. $\{ ax^2 + bx \mid a, b \in \mathbb{R} \}$ ✓.

$\{ f(x) : [0,1] \to \mathbb{R}$ continuous $f(0) = f(1) = 1 \}$ ✗