Week 9 Thursday.

Recall vector space

1. $V = \{ x^n + a_{n-1}x^{n-1} + \dots + a_0 \mid a_i \in \mathbb{R} \}$ ✗

$V = \{ f(x) : [0,1] \to \mathbb{R} \text{ continuous} \mid f(0) + 3f(1) = 0 \}$ ✓

$(f+g)(0) + 3(f+g)(1) = f(0) + 3f(1) + g(0) + 3g(1) = 0$

$f(0) - 3f(1)^2 = 0$

$(C\cdot f)(0) - 3\cdot(Cf)(1)^2$

$= C\cdot f(0) - 3\cdot C^2 \cdot f(1)^2 \neq 0$

$f^2(0) - 2\cdot f(1)^2 = 0$

$(f+g)^2(0) = (f(0)+g(0))^2$

$= f^2(0) + g^2(0) + 2f(0)\cdot g(0)$

$$V = \left\{ \begin{pmatrix} 2s - 2t \\ r+s-2t \\ 4r+s+1 \\ 3r-s-2t \end{pmatrix} \middle| \ r,s,t \in \mathbb{R} \right\}$$

$$r\cdot\underbrace{\begin{pmatrix} 0\\1\\4\\3 \end{pmatrix}}_{=\vec a} + s\cdot\underbrace{\begin{pmatrix} 2\\1\\1\\-1 \end{pmatrix}}_{=\vec b} + t\cdot\underbrace{\begin{pmatrix} -2\\-2\\0\\-2 \end{pmatrix}}_{=\vec c} + \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix} \neq \frac{\vec a + \vec b + \vec c}{5}$$

easier example:

$V = \{ r\cdot\begin{pmatrix}1\\0\end{pmatrix} + \cancel{\begin{pmatrix}0\\1\end{pmatrix}} \}$ ✗ because $\begin{pmatrix}0\\0\end{pmatrix}$ is not in V

$\begin{pmatrix}1\\0\end{pmatrix}$

and. $c\cdot\begin{pmatrix}r\\1\end{pmatrix} = \begin{pmatrix}cr\\c\end{pmatrix}$ $c \neq 1$

$V = \{ (r+1)\cdot\begin{pmatrix}1\\0\end{pmatrix} \} = V = \{ r\begin{pmatrix}1\\0\end{pmatrix} \}$

Linear Independence

V $\vec v_1, \dots, \vec v_n \in V$ is linearly independent:

if $\sum \alpha_i \cdot \vec v_i = 0 \Rightarrow \alpha_i = 0 \ \forall i$.

<u>Basis</u>. V $\{\vec{v}_1, \cdots, \vec{v}_n\}$ forms a basis if

① $\text{span}\{\vec{v}_1, \cdots, \vec{v}_n\} = V$

② if $\vec{v}_1, \cdots, \vec{v}_n$ are linearly independent.

Proposition. Fix V any basis contains the same number of vectors.

Pf. If $\{\vec{e}_1, \cdots, \vec{e}_n\}$ is a basis for V

$\{\vec{d}_1, \cdots, \vec{d}_m\}$ ....

and $m > n$. then. $\vec{e}_i = \sum_{j=1}^{m} \alpha_{ij} \cdot \vec{d}_j$ so. we have

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{pmatrix} \qquad A \cdot \begin{pmatrix} \vec{d}_1 \\ \vdots \\ \vec{d}_m \end{pmatrix} = \begin{pmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_n \end{pmatrix}.$$

A has more colums then rows. so

$$A \cdot \vec{x} = \vec{0}$$

has non-trivial solutions, say $\vec{x}_0 = (\beta_1, \cdots, \beta_m)$ is one such solution.

then.

$$A \cdot \begin{pmatrix} \beta_1 \vec{d}_1 \\ \beta_2 \vec{d}_2 \\ \vdots \\ \beta_m \vec{d}_m \end{pmatrix} = \vec{0} \quad \Rightarrow. \quad \beta_1 \vec{d}_1 + \cdots \beta_m \vec{d}_m = \vec{0}$$

where $\beta_1 \cdots \beta_m$ are not all 0.

Contradicts with $\{\vec{d}_1, \cdots, \vec{d}_m\}$ being a basis.

Examples for basis.

⓪ $\mathbb{R}^n$: standard basis $\vec{e}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow$ i-th position. $i = 1, \cdots, n$.

$$\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = x_1 \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$$

① $M_{n \times m}(\mathbb{R})$:

$$\delta_{ij} = \begin{array}{c c} & \begin{array}{ccc} & j & \end{array} \\ i \rightarrow & \begin{pmatrix} 0 & \vdots & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \ddots \end{pmatrix} \end{array} \quad \text{for } 1 \le i \le n, \; 1 \le j \le m.$$

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{pmatrix} = \sum_{i,j} \alpha_{ij} \cdot \delta_{ij}$$

⑥ $V = \{$ set of linear transformations : $\mathbb{R}^m \longrightarrow \mathbb{R}^n \}$.

$\{\vec{e}_i\}$ $\{\vec{d}_j\}$

$$\begin{cases} \tilde{\delta}_{ij}(\vec{e}_i) = \vec{e}_j \\ \tilde{\delta}_{ij}(\vec{e}_k) = \vec{0} \quad \text{for } k \neq i \end{cases}$$

$$T(\vec{e}_i) = \sum_{1 \le j \le m} \alpha_{ij} \vec{d}_j \qquad T = \sum \alpha_{ij} \cdot \tilde{\delta}_{ij}$$

Check this.

Change of Basis.

$V = \mathbb{R}^2$ $\quad \left\{ \overset{\vec{e}_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \overset{\vec{e}_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right\}$ is a basis

$\left\{ \overset{\vec{d}_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \overset{\vec{d}_2}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \right\}$ is a basis

$$\begin{cases} \vec{d}_1 = \vec{e}_1 \\ \vec{d}_2 = \vec{e}_1 + \vec{e}_2 \end{cases} \quad \text{so.} \quad \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{\bar{\bar{A}}.} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix} = \begin{pmatrix} \vec{d}_1 \\ \vec{d}_2 \end{pmatrix}$$

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\vec{e}_1 \longrightarrow 2\vec{e}_1 + 3\vec{e}_2$

$\vec{e}_2 \longrightarrow 3\vec{e}_1 + 2\vec{e}_2$

Matrix for T under $\{\vec{e}_1, \vec{e}_2\}$ is

$$T_1 = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix} = \begin{pmatrix} T(\vec{e}_1) \\ T(\vec{e}_2) \end{pmatrix}$$

under $\{d_1, d_2\}$ is

$$T_2 = \begin{pmatrix} -1 & 3 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \vec{d}_1 \\ \vec{d}_2 \end{pmatrix} = \begin{pmatrix} T(\vec{d}_1) \\ T(\vec{d}_2) \end{pmatrix}$$

$$\vec{d}_1 = \vec{e}_1 \longrightarrow 2\vec{e}_1 + 3\vec{e}_2 = -\vec{d}_1 + 3\vec{d}_2$$

$$\begin{aligned} \vec{d}_2 = \vec{e}_1 + \vec{e}_2 \longrightarrow\ & 2\vec{e}_1 + 3\vec{e}_2 \\ & + 3\vec{e}_1 + 2\vec{e}_2 \\ & = 5\vec{e}_1 + 5\vec{e}_2 \\ & = 5\vec{d}_2 \end{aligned}$$

$$\boxed{T_1 = A^{-1} T_2 A}$$

Compute A for given basis?

One way is to solve $M \cdot \vec{x} = \vec{d}_j$ for each j

with $M = \begin{pmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{pmatrix}$

To save a little bit time.

$$\left(\begin{array}{ccc|ccc} \vec{e}_1 & \cdots & \vec{e}_n & d_1 & \cdots & \vec{d}_n \end{array}\right)$$