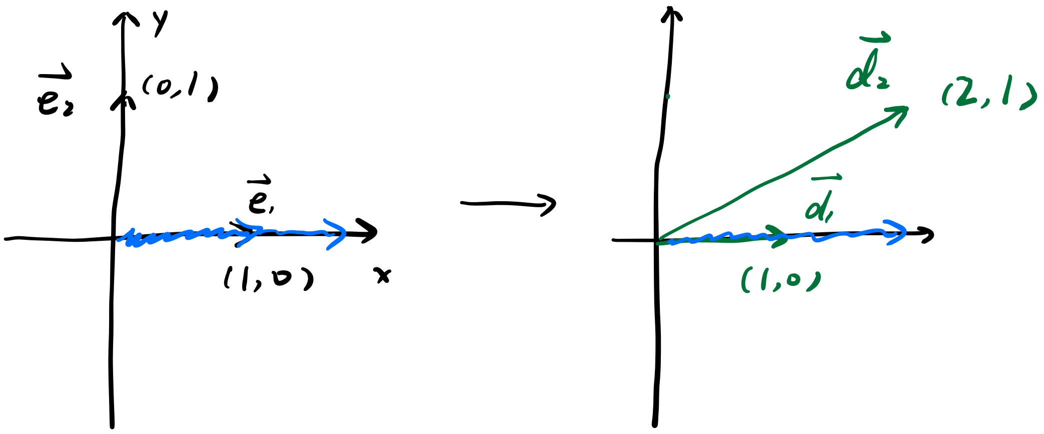
Week 11 Thursday

5.1, 5.2, 5.3

Example:

T:



Matrix for T under $\{\vec{e}_1, \vec{e}_2\}$ $\qquad A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$$T \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$x_1 \cdot \vec{e}_1 + x_2 \cdot \vec{e}_2$

$$T \cdot \vec{e}_1 = \vec{e}_1 \qquad A \cdot \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$$

Def: T: linear transformation $\mathbb{R}^n \to \mathbb{R}^n$

$T\vec{x} = \lambda \cdot \vec{x}$ for some $\vec{x} \neq \vec{0}$. then we say

λ is an eigenvalue for T. $\vec{x}$ is an eigenvector associated to λ.

$T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $\qquad \lambda = 1$ is an eigenvalue.

To find all eigenvectors associated to $\lambda = 1$.

$$T \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \iff (T - I) \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0}$$

$$\iff \begin{pmatrix} 0 & \boxed{2} \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\iff \vec{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = x_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

To find all eigenvalues.

$$T \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \iff (T - \lambda I) \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

has non-zero solution   has non-zero solution

$$\iff T - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{pmatrix}$$

is non invertible.

$*$: A is non invertible $\iff$ $Null(A) \neq \{\vec{0}\}$

$\iff$ $Col(A) \neq \mathbb{R}^n$

$\iff$ $\det(A) = 0$

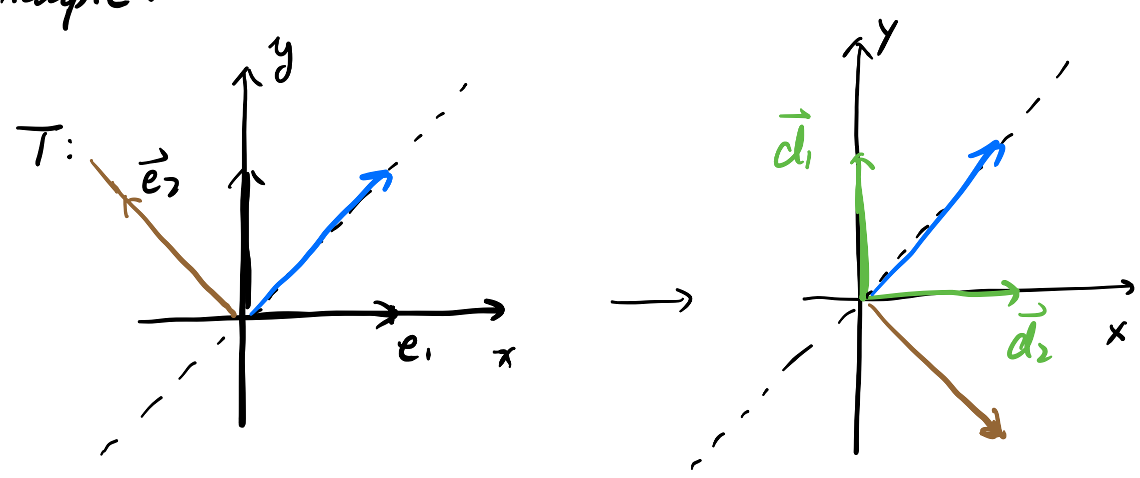
$\iff$ λ is an eigenvalue of A.

$$T \cdot \vec{x} = 0 \cdot \vec{x} = \vec{0}$$

$$\iff (1-\lambda)^2 - 0 = 0$$

$$(\lambda - 1)^2 = 0 \iff \lambda = 1$$

Example:



Matrix for T
under $\{\vec{e}_1, \vec{e}_2\}$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

1). Find all eigenvalues.

$$\underline{\det(A - \lambda I)} = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \underline{\lambda^2 - 1} = 0$$

characteristic polynomial for A.

$\lambda = 1$ or -1

Def: For each eigenvalue λ.
$\{\vec{x} \mid T\vec{x} = \lambda\vec{x}\}$ eigenspace. V_λ

2) Find all eigenvectors.

$\lambda = 1$. solve $A\vec{x} = \vec{x}$ $\iff \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\iff \begin{pmatrix} \boxed{-1} & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\vec{x} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -1$ $A\vec{x} = -\vec{x} \iff \begin{pmatrix} \boxed{1} & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$$\vec{x} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

3). Find matrix for T under the basis

$\vec{d_1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{d_2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

characteristic poly is: $\det(A' - \lambda I)$

$$= \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = \lambda^2 - 1$$

Def. T is diagonalisable if $\exists$ $\underbrace{S = \{\vec{v}_1, \cdots, \vec{v}_n\}}^{\text{a basis for } \mathbb{R}^n}$ s.t.

matrix for T under S is diagonable matrix.

$\Leftrightarrow$ If all eigenvectors span $\mathbb{R}^n$. λ_1 λ_2

$$T(\vec{x}) = \lambda_1 \vec{x}$$
$$\shortparallel \quad \Rightarrow \vec{x} = \vec{0}.$$
$$T(\vec{x}) = \lambda_2 \vec{x}$$

$$\{\vec{x}_1, \cdots, \vec{x}_k, \vec{y}_1, \cdots, \vec{y}_s\}$$

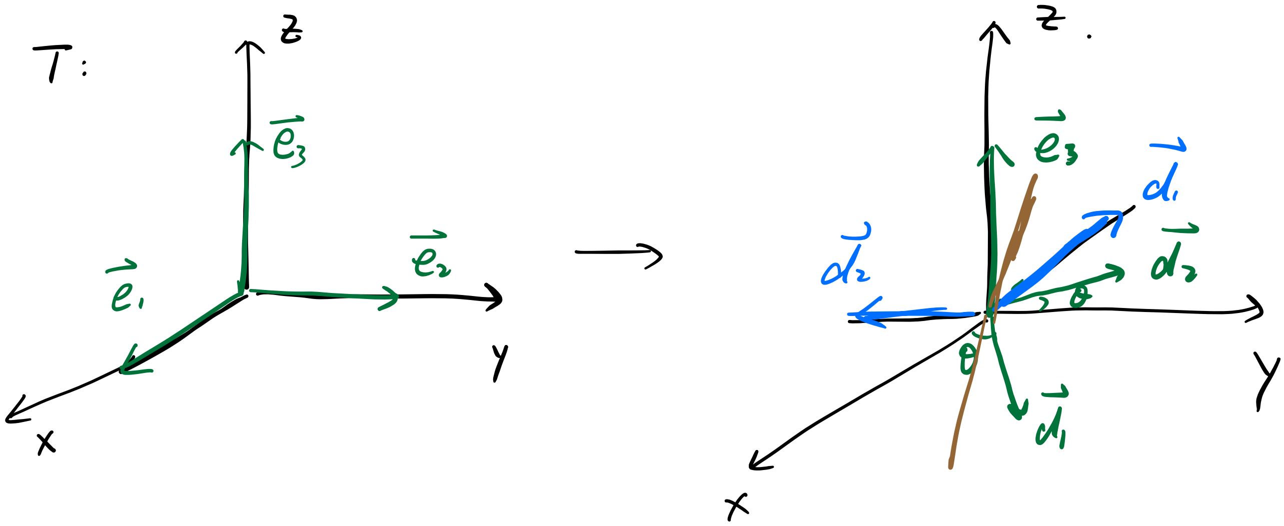
$$\underbrace{\Sigma \alpha_i \vec{x}_i}_{\in V_{\lambda_1}} + \underbrace{\Sigma \beta_i \vec{y}_i}_{} = 0$$

$V_{\lambda_2} \cap V_{\lambda_1} = \vec{0} \Rightarrow \beta_i = 0$

$\alpha_i = 0$

Exaple:

T:



matrix for T $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

1). Find all eigenvalues

$$\det(A-\lambda I)=\begin{vmatrix} \cos\theta-\lambda & -\sin\theta & 0 \\ \sin\theta & \cos\theta-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$=(1-\lambda)\cdot\begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{vmatrix}$$

$$=(1-\lambda)\cdot\left[(\cos\theta-\lambda)^2+\sin^2\theta\right]=0$$

$\lambda=1$ $\quad$ $\sin\theta=0$ $\quad$ $\cos\theta-\lambda=0$

$\Downarrow$

$\cos\theta=\pm1$ $\quad$ $\lambda=\cos\theta$.

$\cos\theta=1 \Rightarrow \theta=0$

$\cos\theta=-1 \Rightarrow \theta=\pi$

If $\theta = 0$ or π then $\lambda = 1$ or $\cos\theta$

If $\theta \neq 0, \pi$ then $\lambda = 1$

2) Determine eigenvectors:

$\lambda = 1$. $A \cdot \vec{x} = \vec{x} \Leftrightarrow \begin{pmatrix} \cos\theta - 1 & -\sin\theta & 0 \\ \sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

since $(\cos\theta - 1)^2 + \sin\theta^2 > 0$ if $\theta \neq 0$ or π.

$$\vec{x} = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

if $\theta = \pi$. ($\theta = 0$ is too boring)

$\lambda = \cos\theta = -1$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \boxed{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = x_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$A' = A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example: $A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$

1) Eigenvalue

2) Eigenvector

3) Matrix A'

1).

$$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 25 - 9 =$$

$$\lambda^2 - 10\lambda + 16$$

$$= (\lambda-2)(\lambda-8) = 0$$

$\lambda = 2$ or 8

2) $\lambda = 2$. $\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0}$ $\quad \vec{x} = x_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = 8$ $\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ $\quad \vec{x} = x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

3) $A' = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$.