Week 12 Tuesday.

Recall eigenvalue, eigenvector, characteristic poly.

eg. $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

1) Find all eigenvalues.

2) Find eigenvectors.

3). Matrix for the linear transformation (corresponding A under standard basis) under the eigenvector basis.

1) $\det \begin{pmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix} = (\lambda-3)^2 - 4 = \lambda^2 - 6\lambda + 5 = 0$

$(\lambda-5)\cdot(\lambda-1) = 0 \qquad \lambda_1 = 1$ and $\lambda_2 = 5$.

2) For $\lambda_1 = 1$ Solve for $A \cdot \vec{x} = \lambda_1 \cdot \vec{x}$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \iff \vec{x} = x_2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = 5$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \iff \vec{x} = x_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

3) Call $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$T(\vec{v}_1) = 1 \cdot \vec{v}_1$

$T(\vec{v}_2) = 5 \cdot \vec{v}_2$

so $A' = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$

Def. (Similar) Given A and $B \in M_{n\times n}(\mathbb{R})$. We say A and B are similar if $\exists$ an invertible matrix C s.t.

$$A = C^{-1} \cdot B \cdot C$$

If A and B are similar, then they share the same char poly.

$$\begin{aligned} \det(A - \lambda I) &= \det(C^{-1} B C - \lambda I) \\ &= \det(C^{-1} \cdot (B - \lambda I) \cdot C) \\ &= \det(C^{-1}) \cdot \det(B - \lambda I) \cdot \det(C) \\ &= \det(B - \lambda I) \end{aligned}$$

$$C^{-1} \cdot \lambda \cdot I \cdot C = \lambda \cdot C \cdot C^{-1} = \lambda \cdot I$$

Change of basis for a linear transformation T

$A \in M_{2\times 2}$ $\quad \vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 = x_1' \vec{v}_1 + x_2' \vec{v}_2$

$$C := \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \\ \downarrow & \downarrow \end{pmatrix}$$

$$C \cdot \underbrace{\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}}_{= \vec{x}'} = \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{= \vec{x}}$$

$$\begin{tikzcd} \vec{x} \arrow[r, "T"] \arrow[d] & A\vec{x} = \vec{y} \arrow[d] \\ C^{-1}\vec{x} = \vec{x}' \arrow[r] & C^{-1} \cdot (A\vec{x}) = \vec{y}' \end{tikzcd}$$

$$C^{-1} \cdot (A\vec{x}) = \boxed{C^{-1} \cdot A \cdot C}\, \vec{x}'$$

eg. $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ $\vec{v_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\vec{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$C = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ $C^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$A' = C^{-1} \cdot A \cdot C = \frac{1}{2} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

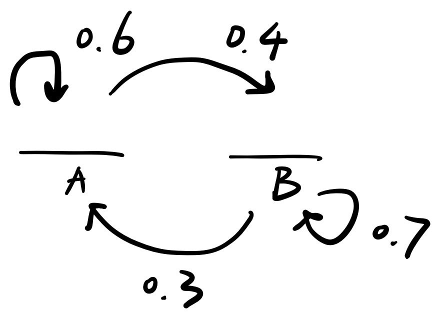
$$= \frac{1}{2} \cdot \begin{pmatrix} 1 & -1 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

Def. (diagonalized) Given a matrix $A \in M_{n \times n}$, if $\exists$ C invertible s.t. $C^{-1}AC$ is diagonal, then we say A is diagonalizable.

<u>Examples</u>:

Markov Chain



A frog has probability

$A = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

For Markov chain matrix 1 is always eigenvalue. $A \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{x}_{k+1} = A \cdot \vec{x}_k$ $\vec{x}_k$: probability the frog is at A and B.

$\vec{x}_0 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ what is $\vec{x}_k$ when $k \to \infty$

(*) Thm. If A has all positive entries, then there is a unique vector $\vec{x}$ s.t. $A\vec{x} = \vec{x}$.

1. Eigenvalues & Eigenvectors.

$$100 \begin{vmatrix} 0.6-\lambda & 0.3 \\ 0.4 & 0.7-\lambda \end{vmatrix} = \begin{vmatrix} 6-\underbrace{10\lambda}_{\tilde{\lambda}} & 3 \\ 4 & 7-10\lambda \end{vmatrix} = \tilde{\lambda}^2 - 13\tilde{\lambda} + 42 - 12$$

$$= \tilde{\lambda}^2 - 13\tilde{\lambda} + 30$$

$$= (\tilde{\lambda} - 10)(\tilde{\lambda} - 3) = 0$$

$\lambda = 1$ or 0.3

For $\lambda = 1$. solve. $\begin{pmatrix} -0.4 & 0.3 \\ 0.4 & -0.3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \iff 4x_1 = 3x_2 \iff \vec{x} = x_2 \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\lambda = 0.3$ solve $\begin{pmatrix} 0.3 & 0.3 \\ 0.4 & 0.4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \iff \vec{x} = x_2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The eigenvectors are $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Now $\vec{x}_0 = x_1' \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} + x_2' \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 3 & 1 & 0.5 \\ 4 & -1 & 0.5 \end{array}\right) \sim \left(\begin{array}{cc|c} 12 & 4 & 2 \\ \cancel{12} & \cancel{-3} \; -7 & \cancel{1.5} \; -0.5 \end{array}\right)$$

$\Rightarrow x_2 = \frac{1}{14}$

$\overset{6}{\cancel{12}} x_1 + \overset{2}{\cancel{4}} \cdot x_2 = \cancel{2} \, 1$

$\Rightarrow x_1 = (1 - \frac{1}{7})/6$

$= \frac{1}{7}$

$$\vec{x}_0 = \frac{1}{7} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \frac{1}{14} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\vec{x}_1 = A(\vec{x}_0) = \frac{1}{7} \cdot A \cdot \vec{v}_1 + \frac{1}{14} \cdot A \cdot \vec{v}_2$$

$$= \frac{1}{7} \cdot \vec{v}_1 + \frac{1}{14} \cdot (0.3) \cdot \vec{v}_2$$

$$\vec{X}_k = A(A^{k-1}(\vec{x}_0)) = A \cdot \left(\frac{1}{7} \cdot \vec{v}_1 + \frac{1}{14} \cdot (0.3)^{k-1} \cdot \vec{v}_2 \right)$$

$$= \frac{1}{7} \cdot \vec{v}_1 + \frac{1}{14} \cdot (0.3)^k \cdot \vec{v}_2$$

So as $k \to \infty$. the frog will stabilize at $\begin{pmatrix} 3/7 \\ 4/7 \end{pmatrix}$.

<u>Dynamical Systems.</u>

(Assume A has real eigenvalue)

$\vec{X}_{k+1} = A \cdot \vec{X}_k$

Q: For what values of p.

we always get $\lim_{k \to \infty} \vec{X}_k = \vec{v}$ for some $\vec{v}$.

(for all $\vec{x}_0$).

$$A = \begin{pmatrix} 0.4 & 0.3 \\ p & 1.2 \end{pmatrix}$$

We need $-1 < \lambda_i \leq 1$ for all i.

$$\vec{x}_0 = x_1 \underset{\lambda_1}{\vec{v}_1} + x_2 \underset{\lambda_2}{\vec{v}_2}$$

$$A^k \vec{x}_0 = x_1 \cdot \lambda_1^k \vec{v}_1 + x_2 \cdot \lambda_2^k \vec{v}_2$$

$$\det(A - \lambda I) = \begin{vmatrix} 0.4-\lambda & 0.3 \\ p & 1.2-\lambda \end{vmatrix} = (\lambda - 0.4)(\lambda - 1.2) - 0.3 \cdot p$$

$$= \lambda^2 - 1.6\lambda + (0.48 - 0.3p)$$

$$\lambda_1 + \lambda_2 = 1.6$$

$$\lambda_1 \cdot \lambda_2 = 0.48 - 0.3p$$

$$\begin{cases} -1 < \lambda \le 1 \\ -1 < 1.6-\lambda \le 1 \end{cases} \iff 0.6 \le \lambda \le 1$$

$$\iff 0.6 \le \lambda \cdot (1.6-\lambda) = -\lambda^2 + 1.6\lambda = -(\lambda - 0.8)^2 + 0.64 \le 0.64.$$

$$\iff 0.6 \le 0.48 - 0.3p \le 0.64$$

$$-0.16 \le 0.3p \le -0.12$$

$$-\frac{8}{15} \le p \le -\frac{2}{5}$$

If $p = \frac{-8}{15}$, then. $\lambda_1 = \lambda_2 = 0.8$ only one eigenvector.

$-\frac{2}{5} \geq p > \frac{-8}{15}$ then $0.6 \le \lambda_1 \neq \lambda_2 \le 1$ ✓

If $-\frac{8}{15} < p \le -\frac{2}{5}$, then we're guaranteed $\vec{x}_k$ converges.

*: We will discuss $p = \frac{-8}{15}$ for next time.