Week 12 Thursday.

Recall. $\vec{x}_{k+1} = A\vec{x}_k$

If $A \in M_{2\times 2}(\mathbb{R})$ and has two different eigen value $\lambda_1 \neq \lambda_2$. and eigenvector $\vec{v}_1$ and $\vec{v}_2$

then. for any vector $\vec{x}_0 = x_1\vec{v}_1 + x_2\vec{v}_2$

$$A^k\vec{x}_0 = x_1 \cdot \lambda_1^k \vec{v}_1 + x_2 \cdot \lambda_2^k \vec{v}_2$$

1. What if A cannot be diagonalized?

eg. $A = \begin{pmatrix} 0.8 & 1 \\ 0 & 0.8 \end{pmatrix}$ $\quad \det(A - \lambda I) = \begin{vmatrix} 0.8-\lambda & 1 \\ 0 & 0.8-\lambda \end{vmatrix} = (\lambda - 0.8)^2 = 0$

$\Rightarrow \lambda_1 = \lambda_2 = 0.8$ $\quad (A - 0.8I)\cdot\vec{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$$\Rightarrow \vec{v} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = x_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$A^k \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda^k \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad$ So if $\vec{x}_0 = \vec{e}_1$ then $\vec{x}_k \to \vec{0}$.

$A\vec{e}_1 = \lambda\vec{e}_1$

$A\vec{e}_2 = \lambda\vec{e}_2 + \vec{e}_1$

$A^2\vec{e}_2 = \lambda \cdot A\vec{e}_2 + A\vec{e}_1 = \lambda \cdot (\lambda\vec{e}_2 + \vec{e}_1) + \lambda\vec{e}_1$

$= \underline{\lambda^2}\,\vec{e}_2 + \underline{2\lambda}\,\vec{e}_1$

$A^3\vec{e}_2 = \lambda^2 \cdot A\vec{e}_2 + 2\lambda A\vec{e}_1 = \lambda^2 \cdot (\lambda\vec{e}_2 + \vec{e}_1) + 2\lambda \cdot \lambda\vec{e}_1$

$= \underline{\lambda^3}\,\vec{e}_2 + \underline{3\lambda^2}\,\vec{e}_1$

$$\lambda^3 \cdot (\lambda \vec{e_2} + \vec{e_1}) + 3\lambda^2 \cdot \lambda e_1$$
$$= \lambda^4 \vec{e_2} + 4\lambda^3 e_1$$

$$A^k \vec{e_2} = \lambda^k \vec{e_2} + k \cdot \lambda^{k-1} \vec{e_1}$$

$\lambda = 0.8$ $\quad \lim_{k\to\infty} \lambda^k = 0 \qquad \lim_{k\to\infty} k \cdot \lambda^{k-1} = 0$

2. Complex eigenvalue.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

$$\lambda^2 + 1 = 0 \iff \lambda = \pm i$$

$\lambda_1 = i$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \quad \vec{x} = \begin{pmatrix} i x_2 \\ x_2 \end{pmatrix} = x_2 \cdot \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$\lambda_2 = -i$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \quad \vec{x} = x_2 \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \quad \text{then} \quad P^{-1} \cdot A \cdot P = \begin{pmatrix} i & \\ & -i \end{pmatrix}$$

Way 1:

$$\begin{array}{cccc} 1 & 0 & -i + \frac{i}{2} = -\frac{i}{2} & \frac{1}{2} \\ 1 & -1 & -i & \end{array}$$

$$\left(\begin{array}{cc|cc} \cancel{i} & \cancel{-i} & \cancel{1} & 0 \\ \cancel{1} & \cancel{1} & \cancel{0} & 1 \end{array}\right)$$

$$\begin{array}{cccc} 0 & \cancel{2} & \frac{i}{2} & \frac{1}{2} \\ & 1 & & \end{array}$$

$$P^{-1} = \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

Way 2: Cramer's $P^{-1} = \frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$

$$= \frac{1}{2} \cdot \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$$

$$P^{-1}AP = \frac{1}{2} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \cdot \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} =: D$$

$A^{100} = ?$

$A = P \cdot D \cdot P^{-1}$ then $A^{100} = (PD\cancel{P^{-1}}) \cdot (\cancel{P}D\cancel{P^{-1}}) \cdot \cdots (\cancel{P}DP^{-1})$

$$= P \cdot D^{100} \cdot P^{-1}$$

$$= P \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot P^{-1} = I.$$

eg. $A = \begin{pmatrix} -1 & -4 \\ 2 & 3 \end{pmatrix}$

1) Find eigenvalue eigenvector.

2). $A^{100} = ?$

1). $\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -4 \\ 2 & 3-\lambda \end{vmatrix} = (\lambda - 3)(\lambda + 1) + 8$

$$= \lambda^2 - 2\lambda + 5$$

$\Delta = 4 - 20 = -16$ $\lambda = \dfrac{2 \pm \sqrt{\Delta}}{2} = 1 \pm 2i$

$\sqrt{\Delta} = 4 \cdot i$ $\begin{cases} \Delta = b^2 - 4ac \\ x = \dfrac{-b \pm \sqrt{\Delta}}{2} \end{cases}$

<u>Rmk</u>: if A is $M_{n\times n}(\mathbb{R})$ and λ is a complex eigvalue with eigenvector $\vec{v}$, i.e. $A\vec{v} = \lambda\vec{v}$, then.

$$\overline{A\cdot\vec{v}} = A\cdot\overline{\vec{v}} = \overline{\lambda\vec{v}} = \overline{\lambda}\cdot\overline{\vec{v}}$$

so $\overline{\lambda}$ is also an eigenvalue of A with eigenvector $\overline{\vec{v}}$.

$\lambda_1 = 1+2i$ $\begin{pmatrix} \underset{\underset{-2-2i}{\parallel}}{-1-(1+2i)} & -4 \\ 2 & \underset{\underset{2-2i}{\parallel}}{3-(1+2i)} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$$\left(\begin{array}{cc|c} \cancel{2}^{\,1} & \cancel{2-2i}^{\,1-i} & 0 \\ \underset{0}{\cancel{-2-2i}} & \underset{0}{\cancel{-4}} & 0 \end{array}\right) \Rightarrow \vec{x} = \begin{pmatrix} -(1-i)\cdot x_2 \\ x_2 \end{pmatrix}$$

$(1-i)(2+2i) = 2\cdot(1-i^2) = 4$ $= x_2 \cdot \begin{pmatrix} i-1 \\ 1 \end{pmatrix}$

$\lambda_2 = 1-2i$ by the rmk. $\vec{x} = x_2 \cdot \begin{pmatrix} -i-1 \\ 1 \end{pmatrix}$ $\overline{a+bi} = a-bi$

2) $D = \begin{pmatrix} 1+2i & 0 \\ 0 & 1-2i \end{pmatrix}$ $P = \begin{pmatrix} i-1 & -i-1 \\ 1 & 1 \end{pmatrix}$

$\det(P) = (i-1)+(i+1) = 2i$

$A = PDP^{-1}$ $P^{-1} = \dfrac{1}{2i}\begin{pmatrix} 1 & i+1 \\ -1 & i-1 \end{pmatrix}$

$\Updownarrow$

$(D = P^{-1}AP)$

$$A^{100} = P\cdot D^{100}\cdot P^{-1} = \frac{1}{2i}\begin{pmatrix} i-1 & -i-1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} (1+2i)^{100} & 0 \\ 0 & (1-2i)^{100} \end{pmatrix}\begin{pmatrix} 1 & i+1 \\ -1 & i-1 \end{pmatrix}$$

$1+2i = \rho \cdot e^{i\theta}$

$(1+2i)^{100} = \rho^{100} \cdot e^{100 i\theta}$

$$= \frac{1}{2i} \cdot \begin{pmatrix} (i-1)\cdot(1+2i)^{100} & (-i-1)(1-2i)^{100} \\ (1+2i)^{100} & (1-2i)^{100} \end{pmatrix} \begin{pmatrix} -1 & i+1 \\ 1 & i-1 \end{pmatrix}$$

$$= \begin{pmatrix} (1+2i)^{100}(1-i) - (1-2i)^{100}(1+i) \big/ 2i & -2\cdot(1+2i)^{100} + 2\cdot(1-2i)^{100} \big/ 2i \\ -(1+2i)^{100} + (1-2i)^{100} \big/ 2i & (1+2i)^{100}\cdot(i+1) + (1-2i)^{100}\cdot(i-1) \big/ 2i \end{pmatrix}$$

3. Differential Equation.

$\vec{X}_{k+1} = A \cdot \vec{X_k}$ $\qquad$ $\vec{X}_{k+1} - \vec{X_k} = (A - I)\cdot \vec{X_k}$

$\vec{X}(t) = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix}$ $\qquad$ $X_i(t): \mathbb{R} \to \mathbb{R}$.

$\vec{x}'(t) = A \cdot \vec{x}(t)$

If $\vec{x}(t) = (x_1(t))$ then $x_1'(t) = a_1 x_1(t)$

$$\frac{d x_1(t)}{dt} = a_1 \cdot x_1(t)$$

$$\int \frac{d x_1(t)}{x_1(t)} = \int a_1 \, dt \qquad \ln|x_1(t)| = a_1 t + C$$

$$x_1(t) = C \cdot e^{a_1 t}.$$

If A is diagonal matrix.

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ \vdots \\ x_n(t) \end{pmatrix}$$

$\vec{x_i}'(t) = a_i \cdot x_i(t)$

If A is diagonalizable. say $A = PDP^{-1}$. then.

take linear transformation on variables $\vec{y}(t) = P^{-1} \cdot \vec{x}(t)$

then $\vec{y}'(t) = D \cdot \vec{y}(t)$ which is diagonal.

solve for $\vec{y}$ and the $\vec{x}(t) = P \cdot \vec{y}(t)$

eg. $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$ $\quad \vec{x}'(t) = A \cdot \vec{x}(t)$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{vmatrix} = \lambda^2 - 4 + 3 = \lambda^2 - 1 = 0$$

$\lambda = \pm 1$.

$\lambda = 1$ $\quad \begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \iff \vec{x} = \begin{pmatrix} -3x_2 \\ x_2 \end{pmatrix} = x_2 \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\lambda = -1$ $\quad \begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \iff \vec{x} = x_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

so $P = \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}$ $\quad P^{-1} = \frac{1}{-2} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}$

$\vec{y}(t) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix} \vec{x}(t)$ $\quad$ i.e. $\quad y_1(t) = \frac{-1}{2} x_1(t) - \frac{1}{2} x_2(t)$

$\quad y_2(t) = \frac{1}{2} x_1(t) + \frac{3}{2} x_2(t)$

then we get. $\vec{y}'(t) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \vec{y}(t)$