Week 14 Tuesday.

Recall. a basis S for V is

1) linearly independent

2) $\operatorname{span}(S) = V$

Q: Given V, $S = \langle \vec{e}_1, \cdots, \vec{e}_n \rangle$. $\vec{b} \in V$.

$$A = \begin{pmatrix} \downarrow & \downarrow & \cdots & \downarrow \\ \vec{e}_1 & \vec{e}_2 & & \vec{e}_n \end{pmatrix} \qquad A \cdot \underset{\text{coefficients.}}{\underset{\nearrow}{\vec{x}}} = \vec{b}.$$

$$V = \mathbb{R}^n \qquad S = \left\langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \cdots \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \right\rangle. \qquad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n.$$

$\vec{x} = \vec{b}$.

Orthonormal basis for $\mathbb{R}^n$.

$S = \{ \vec{v}_1, \cdots, \vec{v}_n \}$

S is orthonomal basis if

1) S is basis

2) $\vec{v}_i \cdot \vec{v}_j = 0 \quad (\Leftrightarrow \vec{v}_i \perp \vec{v}_j) \quad$ if $i \neq j$

3) $\vec{v}_i \cdot \vec{v}_i = 1$

Suppose S is orthonomal. and $\vec{b} = \sum_i b_i \cdot \vec{v}_i$

$$\vec{b} \cdot \vec{v}_j = \left(\sum_{i=1}^{n} b_i \vec{v}_i \right) \cdot \vec{v}_j = \sum_{i=1}^{n} b_i \cdot \vec{v}_i \cdot \vec{v}_j = b_j \, \vec{v}_j \cdot \vec{v}_j = b_j \cdot \| v_j \|^2$$

Now. $C = \begin{pmatrix} \longrightarrow \vec{v}_1 \\ \longrightarrow \vec{v}_2 \\ \vdots \\ \longrightarrow \vec{v}_n \end{pmatrix}$ $\quad C^T = \begin{pmatrix} \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ \vec{v}_1 & \vec{v}_2 & & & \vec{v}_n \end{pmatrix}$

$$C \cdot C^T = \begin{pmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 & \cdots & \vec{v}_1 \cdot \vec{v}_n \\ \vdots & \ddots & & \vdots \\ & & \vec{v}_i \cdot \vec{v}_j & \\ \vdots & & \ddots & \vdots \\ \vec{v}_n \cdot \vec{v}_1 & \cdots & \cdots & \vec{v}_n \cdot \vec{v}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & \cdots & 1 & 0 & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} = I.$$

Def (Orthogonal Matrix) If $A \in M_{n \times n}(\mathbb{R})$

we say A is an orthogonal matrix if $A \cdot A^T = I$.

$$A \cdot \vec{x} = \vec{b} \iff \vec{x} = A^{-1}\vec{b} = A^T \cdot \vec{b}$$

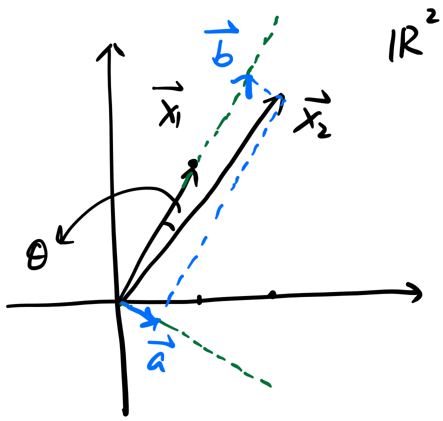
Computational Task:

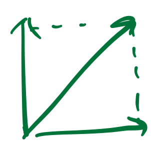
Given $\vec{x}_1, \vec{x}_2, \cdots \vec{x}_n \in \mathbb{R}^n$ (not necessarily orthonormal).

construct an orthonormal basis $\vec{v}_1, \cdots \vec{v}_n$?

eg. $V = \mathbb{R}^n$

$\vec{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \vec{x}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$





$\dfrac{\vec{x}_1}{\|\vec{x}_1\|}$: length 1 , unit vector

$$\vec{x}_2 \cdot \vec{x}_1 = \|\vec{x}_2\| \cdot \|\vec{x}_1\| \cdot \cos\theta$$

$\vec{b}$ is the projection of $\vec{x}_2$ along $\vec{x}_1$.

$$\|\vec{b}\| = \|\vec{x}_2\|\cdot\cos\theta = \frac{\vec{x}_2\cdot\vec{x}_1}{\|\vec{x}_1\|}$$

$$\vec{b} = \|\vec{b}\|\cdot\frac{\vec{x}_1}{\|\vec{x}_1\|} = \left(\frac{\vec{x}_2\cdot\vec{x}_1}{\|\vec{x}_1\|\cdot\|\vec{x}_1\|}\right)\vec{x}_1$$

$$= \left(\frac{\vec{x}_2\cdot\vec{x}_1}{\vec{x}_1\cdot\vec{x}_1}\right)\cdot\vec{x}_1$$

$$\vec{a} = \vec{x}_2 - \left(\frac{\vec{x}_2\cdot\vec{x}_1}{\vec{x}_1\cdot\vec{x}_1}\right)\vec{x}_1 \quad \perp \quad \vec{x}_1.$$

Plug in: $\vec{b} = \left(\frac{8}{5}\right)\cdot\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}8/5\\16/5\end{pmatrix}$

$$\vec{x}_2\cdot\vec{x}_1 = \begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix} = 1\times 2 + 2\times 3 = 8$$

$$\vec{x}_1\cdot\vec{x}_1 = \begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix} = 5$$

$$\vec{a} = \begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}8/5\\16/5\end{pmatrix} = \begin{pmatrix}2/5\\-1/5\end{pmatrix} \qquad \|\vec{a}\| = \frac{1}{\sqrt{5}}$$

$$\vec{x}_1 = \begin{pmatrix}1\\2\end{pmatrix} \qquad \vec{w}_1 = \frac{1}{\sqrt{5}}\cdot\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{pmatrix}$$

$$\vec{a} = \begin{pmatrix}2/5\\-1/5\end{pmatrix} \qquad \vec{w}_2 = \begin{pmatrix}2/\sqrt{5}\\-1/\sqrt{5}\end{pmatrix}$$

$\{\vec{w}_1, \vec{w}_2\}$ is an orthonormal basis.

$\vec{r} = \begin{pmatrix}21\\13\end{pmatrix}$ what is coefficients of $\vec{r}$ in $\{\vec{w}_1, \vec{w}_2\}$.

$$A = \begin{array}{cc} \vec{w}_1 & \vec{w}_2 \\ \end{array}$$
$$A = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix} \qquad A \cdot \vec{x} = \vec{r} \qquad \vec{x} = A^{-1}\,\vec{r} = A^{T} \cdot \vec{r}$$

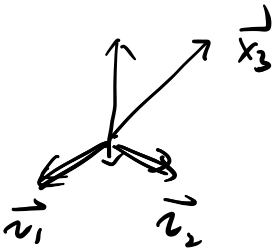
$$A^{T} = \begin{pmatrix} \longrightarrow \vec{w}_1 \\ \longrightarrow \vec{w}_2 \end{pmatrix}$$

$$r_1 = \vec{v}_1 \cdot \vec{r} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \cdot \begin{pmatrix} 21 \\ 13 \end{pmatrix} = \frac{47}{\sqrt{5}}$$

$$r_2 = \vec{v}_2 \cdot \vec{r} = \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix} \cdot \begin{pmatrix} 21 \\ 13 \end{pmatrix} = 29/\sqrt{5}$$

In general, this is called Gram-Schmidt process

$\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n \in \mathbb{R}^m$. $(m > n)$



$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right) \vec{v}_1$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \cdot \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \cdot \vec{v}_2$$

$$\vdots$$

$$\vec{v}_n = \vec{x}_n - \frac{\vec{x}_n \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \cdot \vec{v}_1 - \cdots - \frac{\vec{x}_n \cdot \vec{v}_{n-1}}{\vec{v}_{n-1} \cdot \vec{v}_{n-1}} \cdot \vec{v}_{n-1}$$

Normalize each $\vec{v}_i$ by $\dfrac{\vec{v}_i}{\|\vec{v}_i\|}$, we get orthonormal basis.

Consequence of Gram-Schmidt. : QR factorization

$$\underbrace{\begin{pmatrix} \downarrow \vec{x}_1 & \downarrow \vec{x}_2 \end{pmatrix}}_{\substack{\text{"} \\ \text{general} \\ \text{invertible}}} = \begin{pmatrix} v_1 & v_2 \\ \downarrow & \downarrow \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} v_1/\|\vec{v}_1\| \downarrow & v_2/\|\vec{v}_2\| \downarrow \end{pmatrix}}_{\text{orthogonal}} \cdot \underbrace{\begin{pmatrix} \|\vec{v}_1\| & \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \cdot \|\vec{v}_1\| \\ 0 & \|\vec{v}_2\| \end{pmatrix}}_{\text{upper triangular}}$$

matrix $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ matrix

eg.
$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{5} & 8/\sqrt{5} \\ 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$$

Ex. $\mathbb{R}^4$

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \vec{x}_2 = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{X}_3 = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

Gram-Schmidt process:

$$\vec{x}_3 \cdot \vec{v}_2 = -1 - 6 + \tfrac{1}{2} = -13/2$$

$$\vec{v}_2 \cdot \vec{v}_2 = 1/4 + 4 + 1 + 1/4$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \cdot \vec{v}_1$$

$$= \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -2 \\ 1 \\ 1/2 \end{pmatrix}$$

$$\vec{v}_3 = \vec{X}_3 - \frac{\vec{X}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \cdot \vec{v}_1 - \frac{\vec{X}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \cdot \vec{v}_2$$

$$= \begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{-13/2}{11/2} \cdot \begin{pmatrix} -1/2 \\ -2 \\ 1 \\ 1/2 \end{pmatrix}$$

$$\stackrel{\dots}{=} \begin{pmatrix} 2 - 3/2 - \frac{13}{22} \\ 3 - 26/11 \\ 13/11 \\ 1 - 3/2 + \frac{13}{22} \end{pmatrix} = \begin{pmatrix} -1/11 \\ 7/11 \\ 13/11 \\ 1/11 \end{pmatrix}$$