08/26/21 Week 2 Thursday.

Q1: Is echelon form unique or not?

Q2: Are there linear systems with same solution set. but in different. shape?

$$\begin{cases} x_2 + 2x_3 + x_4 = 0 & ① \\ x_1 - x_2 + x_4 = 1 & ② \\ 2x_1 - x_2 + 2x_3 + 3x_4 = 2 & ③ \end{cases}$$

By row operations. solution set of the old system is contained in the solution set of the new system. Vice versa due to that. row operations are <u>reversible</u>.

eg ③ ⇝ ③ + ② × 1 ✓

② ⇝ ② × 1 ✗

$$\left(\begin{array}{cccc|c} 0 & 1 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 & 1 \\ 2 & -1 & 2 & 3 & 2 \end{array}\right)$$

Switch ① and ③.

$$\begin{pmatrix} 2 & -1 & 2 & 3 & 2 \\ 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}$$

② ⇝ ② − ① × $\frac{1}{2}$

$$\begin{pmatrix} 2 & -1 & 2 & 3 & 2 \\ 0 & -\frac{1}{2} & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}$$

③ ⇝ ③ + ② × 2

$$\begin{pmatrix} 2 & -1 & 2 & 3 & 2 \\ 0 & \underline{-\frac{1}{2}} & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Last time we got.

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & \boxed{1} & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

① ⇝ ① + ②

$$\begin{pmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

free variables.

Reduced Echelon Form

② ⇝ ② × (−2).

$$\begin{pmatrix} 2 & -1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

① ⟶ ① × ½

$$\begin{pmatrix} 1 & -\frac{1}{2} & 1 & \frac{3}{2} & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

① ⟶ ① + ② × $\frac{1}{2}$

$$\begin{pmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

An echelon form is called a reduced echelon form if.

1. Every leading entry is 1
2. Each leading 1 is the only nonzero term in its column.

Thm. For every matrix M. there exists a unique reduced echelon form for M.

$$\left(\begin{array}{cccc|c} \boxed{\times} & \times & \times & \times & \times \\ 0 & 0 & \boxed{\times} & \times & \times \\ 0 & 0 & 0 & \boxed{\times} & \times \end{array}\right)$$

$\uparrow$

free variable.

Columns without pivots are free variables

$$\begin{vmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \\ & \vdots & & \\ & 0 & & \end{vmatrix}$$

$$\begin{pmatrix} 1 & \times & \times & \times & \cdots \\ & & 1 & \times & \cdots \\ & & & & \boxed{1\times} \end{pmatrix}$$

1. Thm. (Existence, uniqueness) A linear system has at least one solution if and only if the rightmost column of the echelon form is not a pivot. (or equivalently there is no pivot in the last column).

2. If there is no free variables. (or equivalently column 1 to column $n-1$ all contain pivot). then the solution is unique. Otherwise, there are infinitely many.

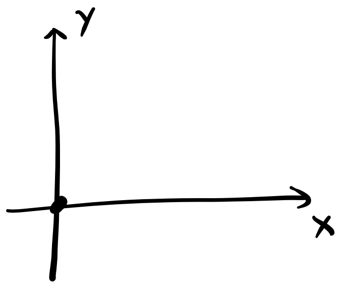
Vectors.

Def 1. A vector is a number + a direction.

Def 2. A column vector is a $n \times 1$ matrix.

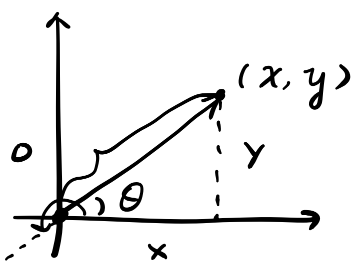
$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

vectors are models for force, velocity, displacement.



number (length) $= \sqrt{x^2+y^2}$

$$\theta = \begin{cases} \arctan \frac{y}{x} & x > 0 \\ \arctan \frac{y}{x} + \pi & x < 0 \\ \frac{\pi}{2} & x = 0,\ y > 0 \\ \frac{3\pi}{2} & x = 0,\ y < 0 \\ \vec{0} & x = 0,\ y = 0 \end{cases}$$



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Operations:

- $\vec{u}+\vec{v}$ $\quad \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{pmatrix}$

- scalor multiplication:

  $a \cdot \vec{u} = a \cdot \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} a \cdot u_1 \\ a \cdot u_2 \\ \vdots \\ a \cdot u_n \end{pmatrix}$

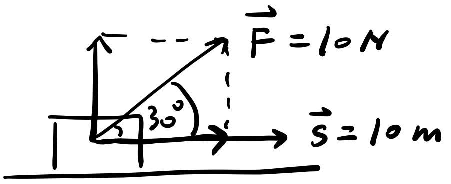
- $\vec{u}-\vec{v} = \vec{u} + (-1) \cdot \vec{v}$

- dot product:

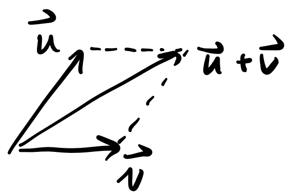
$$\vec{u} \cdot \vec{v} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n \cdot v_n$$ ← a number.

- length (magnitude).

  $\|\vec{u}\| = \sqrt{u_1^2 + \cdots + u_n^2}$



$W = 10^{N} \times \cos 30°$ $\quad$ J
$\times 10m$
$= \vec{F} \cdot \vec{S} = \|\vec{F}\| \cdot \|\vec{S}\| \cdot \cos\theta$



Def. (Linear Combination)

Given $\vec{v}_1, \cdots, \vec{v}_m$ in $\mathbb{R}^n$ $\quad \alpha_1, \cdots, \alpha_m$ in $\mathbb{R}$

$\vec{y} = \sum_i \alpha_i \cdot \vec{v}_i$ is a linear combination of $\vec{v}_1, \cdots, \vec{v}_m$

The set of all linear combinations is called the span of $\{\vec{v}_1, \cdots, \vec{v}_m\}$.