Week 15 Tuesday.

Recall that.

$A \in M_{n\times n}(\mathbb{R})$

In order to diagonalize A, we solve

1) $\det(A-\lambda I)=0$ $\rightsquigarrow$ eigenvalue

2) For each eigenvalue λ, solve $(A-\lambda I)\cdot\vec{x}=0$.

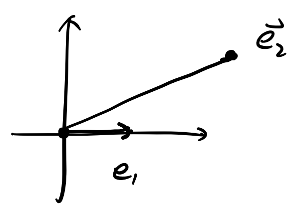
$\rightsquigarrow$ eigenvector(s) $V_\lambda := \{\vec{v} \mid A\vec{v}=\lambda v\}$.

If $\sum_\lambda \dim(V_\lambda)=n$, then we can find a basis $S=\{\vec{v}_1, \cdots \vec{v}_n\}$ of $\mathbb{R}^n$. that are all eigenvectors. $\Rightarrow P^{-1}AP=D$

$P=\begin{pmatrix} v_1 & & v_n \\ \downarrow & \cdots & \downarrow \end{pmatrix}$

Not always diagonalizable:

eg. $A:\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $\mathbb{R}^2$



For symmetric matrix, such bad situation never happens.

Thm. $A\in M_{n\times n}(\mathbb{R})$ $A=A^T$. then A can always be diagonalized.

Rmk. Recall. Last time: if $\lambda_1\neq\lambda_2$ are different eigenvalue. then V_{λ_1} and V_{λ_2} are orthogonal to each other.

So. if A can be diagnalized, then A can also be orthogonal diagonalized.

Prep: Block Matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a\cdot a' + bc & a\cdot b' + b\cdot d' \\ c\cdot a' + d\cdot c' & c\cdot b' + d\cdot d' \end{pmatrix}$$

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right) = \begin{pmatrix} A\cdot A' + B\cdot C' & A\cdot B' + B\cdot D' \\ C\cdot A' + D\cdot C' & C\cdot B' + D\cdot D' \end{pmatrix}$$

Pf. Firstly, we can always find at least one eigenvalue λ_1 for A. then. solve $(A - \lambda_1 I)\cdot\vec{x} = 0$. so we can find at least one eigenvector $\vec{v}_1$ for A.

Assme $\vec{v}_1$ is normalized, and extend it to an orthonormal basis. $\{\vec{v}_1, \cdots, \vec{v}_n\}$. take $P = \begin{pmatrix} \downarrow & \downarrow & \downarrow \\ v_1 & \cdots & v_n \end{pmatrix}$

(use Gram-Schmidt)

$$P^{-1}\cdot A\cdot P = \left(\begin{array}{c|c} \lambda_1 & A_1 \\ \hline 0 & \\ 0 & A_2 \\ 0 & \end{array}\right) \rightsquigarrow \begin{pmatrix} \lambda_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

Since A is symmetric, P is orthogonal matrix. $(P\cdot P^T = I)$

$$(P^{-1}AP)^T = P^T\cdot A^T (P^{-1})^T = P^{-1}\cdot A\cdot P \quad \text{is symmetric}$$

$\Rightarrow \quad A_1 = (0\ 0\ 0) \qquad A_2 = A_2^T$.

Use induction: ① For $n=1$. A is scalar. always. true.

② Assme for $k < n$ $A \in M_{k\times k}(\mathbb{R})$ is always digonalizable

Now. by ②. we can pick $Q \in M_{(n-1)\times(n-1)}$ orthonormal. s.t.

$Q^{-1}\cdot A_2\cdot Q = D$ is diag

Therefore. $\tilde{Q} := \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q \end{array}\right)$ $\quad \tilde{Q}^{-1} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q^{-1} \end{array}\right)$

we can get $\tilde{Q}^{-1} \cdot \left(\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & A_2 \end{array}\right) \cdot \tilde{Q}$

$$= \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q^{-1} \end{array}\right) \left(\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & A_2 \end{array}\right) \cdot \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q \end{array}\right)$$

$$= \left(\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & Q^{-1} A_2 Q \end{array}\right) = \left(\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & \underset{\overset{\parallel}{D}}{\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_{n-1} \end{bmatrix}} \end{array}\right)$$

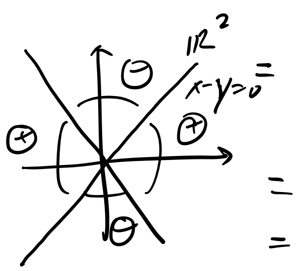
$\square$.

2. Quadratic Form.

$$f(x,y) = x^2 + 2xy - 3y^2 \xrightarrow{/y^2} \tilde{x} = x/y$$

$$\tilde{x}^2 + 2\tilde{x} - 3$$

$$= (\tilde{x} + \alpha)^2 + \beta$$



$$= \underbrace{x^2 + 2xy + y}{}^2 - y^2 - 3y^2$$

$$\underbrace{\tilde{x}^2 + 2\tilde{x} + 1}{} - 1 - 3$$

$$= (\tilde{x}+1)^2 - 4$$

$$= (x+y)^2 - 4y^2$$

$$= [(x+y)+2y] \cdot [(x+y)-2y]$$

$$g(x,y) = x \cdot y = g_1(x,y)^2 \pm g_2(x,y)^2$$

$$(\tilde{x}+1)^2 - 4$$

$$\frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2$$

$$ax^2 + bxy + cy^2 = \overset{?}{g}_1(x,y)^2 \pm \overset{?}{g}_2(x,y)^2$$

$$f(x,y) = ax^2 + 2bxy + cy^2 \overset{??}{=} (x \quad y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{x}^T \cdot A \cdot \vec{x}$$

$$= (ax+by \quad bx+cy) \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= ax^2 + bxy + bxy + cy^2$$

We see a bijection between.

homogeneous quadratic polynomials $\longleftrightarrow$ symmetric metrices

complete the square $\longleftrightarrow$ A is digonal

In general if $P^{-1}AP = D$ then $A = PD \cdot P^{-1}$

then $\vec{x}^T \cdot A \cdot \vec{x} = \underline{\vec{x}^T \cdot P} \cdot D \cdot \underline{P^{-1} \cdot \vec{x}}$

then $\vec{y} = P^{-1} \cdot \vec{x}$ linear transformation of $\begin{pmatrix} x \\ y \end{pmatrix}$

$$= \begin{pmatrix} P_1 x + P_2 y \\ P_3 x + P_4 y \end{pmatrix}$$

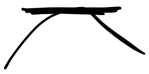
Recall calculus

$$f(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0) \cdot (x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$$+ \Big[\frac{\partial^2 f}{\partial x^2}(x_0,y_0) \cdot \frac{(x-x_0)^2}{2} + \frac{\partial^2 f}{\partial y^2}(x_0,y_0) \cdot \frac{(y-y_0)^2}{2}$$

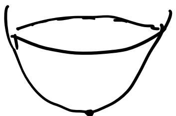
$$+ \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) \cdot \frac{(x-x_0) \cdot (y-y_0)}{2} \Big] + \cdots$$

$$f(x) = f(x_0) + \frac{df}{dx}(x_0) \cdot (x - x_0) + \frac{d^2f}{dx^2}(x_0) \frac{(x-x_0)^2}{2} + \cdots$$



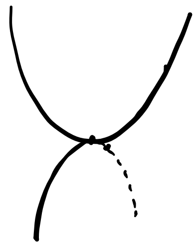
$f''(x_0) > 0$

$f''(x_0) < 0$



$f(x,y)$

x_0, y_0



local min — positive det.

local max — negative det.

saddle — indefinite

Def: $Q(x,y) = ax^2 + 2bxy + cy^2$ then.

if $Q(x,y) > 0$ for every $(x,y) \neq (0,0)$ => positive definite

$Q(x,y) < 0$ . . . . => negative definite

$Q(x,y)$ can be both positive and negative => indefinite