Week 14 Thursday

Dec 9 12:00

Recall symmetric matrix / quadratic form

homogeneous quadratic fuctions

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$\vec{x}^T \cdot A \cdot \vec{x}$$

$$f(x_1, \cdots, x_n) = \sum_{(i,j)} a_{ij} x_i \cdot x_j$$

$$= \sum_{i<j} 2 a_{ij} x_i \cdot x_j + a_{ii} x_i^2$$

$a_{ij} = a_{ji}$

orthogonal digonalize A

complete the square where the linear transformation is "orthogonal" to each other.

Reduce to study diagonalized quadratic form

$$f(x_1, \cdots, x_n) = \lambda_1 g_1(x_1, \cdots, x_n)^2 + \lambda_2 \cdot g_2(x_1, \cdots, x_n)^2 + \cdots + \lambda_n \cdot g_n(x_1 \cdots x_n)^2$$

g_i : linear fuction. degree 1 homogeneous.

λ_i : eigenvalue of A.

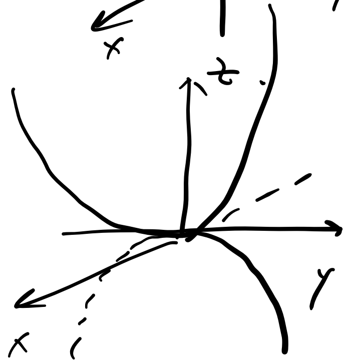
$f(x_1, \cdots, x_n)$ can have different patterns:

- f can be > 0 $(\vec{x} \neq \vec{0})$ positive definite
- f can be < 0 $(\vec{x} \neq \vec{0})$ negative definite.
- f can be both > 0 and < 0 $(\vec{x} \neq 0)$ indefinit.

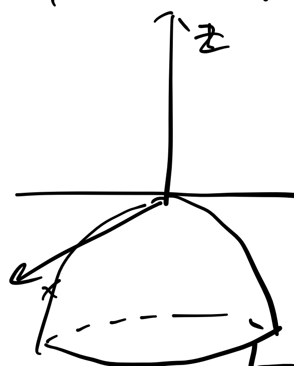
exhaustive classification when A is invertible.

eg. 1) $f(x,y) = x^2 + y^2 \quad \rightsquigarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2) $f(x,y) = 1 \cdot x^2 + (-1) \cdot y^2 \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



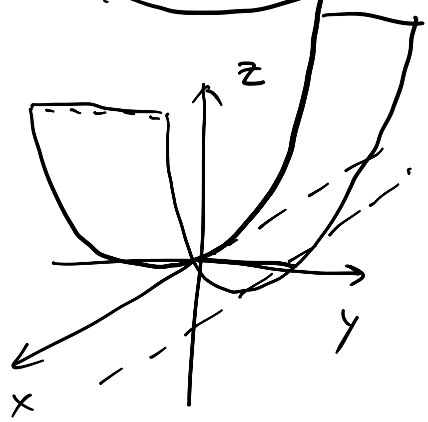
3) $f(x,y) = (-1) \cdot x^2 + (-1) \cdot y^2 \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$



4) $f(x,y) = x \cdot y \qquad \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$

$\overset{diag}{\rightsquigarrow}$ 2)

5) $f(x,y) = 1 \cdot x^2 + 0 \cdot y^2 \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$



not belong to any of the previous types

If f is not diagonalized, then how to tell which type f belongs to?

- solve for eigenvalue
- dim = 2. quick way:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \det(A - \lambda I) = (a-\lambda)\cdot(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

$$= (\lambda - \lambda_1)\cdot(\lambda - \lambda_2)$$

then. $\lambda_1 + \lambda_2 = a+d \leftarrow$ trace of matrix.

$\lambda_1 \cdot \lambda_2 = ad-bc \leftarrow$ determinant of matrix.

For any invertible P.

$\lambda_1\lambda_2\lambda_3$

$\det(P^{-1}AP - \lambda I)$

$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$

$= \det(P^{-1}(A-\lambda I)\cdot P) = \det(A - \lambda I)$

$\lambda_1 + \lambda_2 + \lambda_3$

- $tr(A) > 0$ $\det(A) > 0$ $\Rightarrow$ $\lambda_1 > 0$ $\lambda_2 > 0$ $\Rightarrow$ positive definite
- $tr(A) < 0$ $\det(A) > 0$ $\Rightarrow$ $\lambda_1 < 0$ $\lambda_2 < 0$ $\Rightarrow$ negative definite
- $\det(A) < 0$. $\Rightarrow$ indefinite
- $\det(A) = 0$? Not Sure.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ indefinite

eg. $f(x,y) = 2x^2 + 6xy - 6y^2$ Type?

indefinite. $A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$ $tr(A) = -4 < 0$, $\det(A) = -21 < 0$

$-4x^2 + 6xy - y^2$ $\begin{pmatrix} -4 & 3 \\ 3 & -1 \end{pmatrix}$ $\det = -5 < 0$

indefinite.

$f(x,y,z) = 9x^2 + 7y^2 + 11z^2 - 8xy + 8xz$

$$\begin{pmatrix} 9-\lambda & -4 & 4 \\ -4 & 7-\lambda & 0 \\ 4 & 0 & 11-\lambda \end{pmatrix}$$

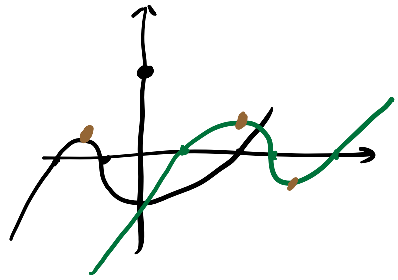
$tr = 27$

$\det = 4\cdot(-28) + 11\cdot(63-16)$

$= -112 + 11\times 47 > 0$

$f(\lambda)$

$$-\det(A-\lambda I) = -\Big(4\cdot[-4\cdot(7-\lambda)] + (11-\lambda)\cdot[(\lambda-9)(\lambda-7)-16]\Big)$$



$$= \lambda^3 - tr\cdot\lambda^2 + \square\cdot\lambda - \det$$

$f(0) = -\det$

$tr(A)>0$ $\quad \det(A)>0$ $\quad \not\Rightarrow$. positive definite.

$\det(A)=\lambda_1\lambda_2\cdot\lambda_3>0 \Rightarrow \begin{cases}\text{either all positive} \\ \text{two negative 1 positive.}\end{cases}$

Last topic Singular Value Decomposition

Due. (next next Tuesday).