Review :

1) Linear Systems $\begin{cases} \text{homogeneous} & A\vec{x}=\vec{0} & S \\ \text{inhomogeneous} & A\vec{x}=\vec{b} & \vec{x}_0+S \end{cases}$

call $\vec{a}_i$: colum vector in A $\in \mathbb{R}^m$

$A\vec{x}=\vec{b}$ $\quad A\in M_{m\times n}(\mathbb{R})$

$\Leftrightarrow x_1\cdot\vec{a}_1+x_2\vec{a}_2+\cdots+x_n\vec{a}_n=\vec{b}$

- consistent ? existence $\leftrightsquigarrow$ in a.m. echelon form no pivot in the last colum

$\Leftrightarrow \vec{b}\in Col(A)$

- how many ? uniqueness $\leftrightsquigarrow$ every column has a pivot

$\Leftrightarrow \vec{a}_1,\cdots,\vec{a}_n$ linearly independent (except last column in a.c)

$A\vec{x}=\vec{b}=A\vec{z} \Leftrightarrow A\cdot(\vec{x}-\vec{z})=\vec{0}$

$\Leftrightarrow dim(Col(A))=n\le m$

$\Leftrightarrow \vec{a}_1,\cdots,\vec{a}_n$ is a basis for $Col(A)$

$\Leftrightarrow Null(A)=\{\vec{0}\}$

$\quad$ " $Ker(A)=\{\vec{x}\mid A\vec{x}=0\}$

$Col(A)\subseteq\mathbb{R}^m$

Algorithm : ① . Echelon Form . (Elementary Row Operation)

Swap . Scalar . Replace

② $Null(A)$: parametric vector form for solutions of $A\vec{x}=\vec{0}$.

$Col(A)$: $A^T \to$ row echelon form $\to E^T$

E

When $n=m$. A is square.

uniqueness of $A\vec{x}=\vec{b}$ $\Leftrightarrow dim(Col(A))=n$

$\Leftrightarrow Col(A)$ span $\mathbb{R}^n$

$A\vec{x}=\vec{b}$. $\quad \Leftrightarrow \exists X.\quad A\cdot X=I$.

let $\vec{b}=\vec{e}_1,\vec{e}_2,\cdots,\vec{e}_n$ $\quad \Leftrightarrow A$ is invertible.

$\vec{x}_1,\vec{x}_2,\cdots,\vec{x}_n$ $\quad \Leftrightarrow det(A)\neq 0$.

Algorithm : ③ $\det(A) =$ dim 2 $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{\det} ad - bc$

dim 3 $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \xrightarrow{\det}$

$a \cdot e \cdot i + bfg + cdh$

$-(afh + bdi + ceg)$

in general $A = R_1 \cdots R_\ell \cdot E$

$= \det(A) = \det(R_1) \cdots \det(R_\ell) \cdot \det(E)$

swap -1

Replace 1

scalar $\times a$ $\times a$

$= \sum_j (-1)^{i+j} a_{ij} \cdot m_{ij}$ Fix i

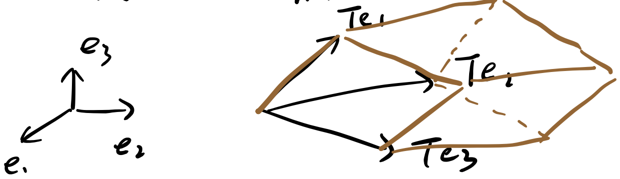
m_{ij} : det (A deleting i-row, j-colm)

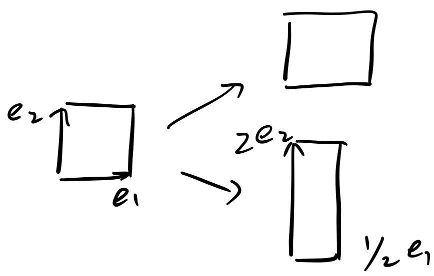
$= \sum_i (-1)^{i+j} a_{ij} \cdot m_{ij}$ Fix j.

A can be considered as the matrix of a linear transformation

T w.r.t standard basis.

$\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$





Algorithm: ③':

A^{-1} = ①

$\left(A \mid I \right) \xrightarrow{\text{row}} \left(I \mid A^{-1} \right)$

②

$\frac{1}{\det(A)} \left((-1)^{i+j} m_{ji} \right)$

To describe "shape" we need more quantities beyond determinant.

2). Eigen value & Eigen vectors.

idea: standard & simple form of a linear transformation

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} \in M_{n\times n}(\mathbb{R}).$$

$\det(D) = d_1 \cdots d_n$

$tr(D) = d_1 + \cdots + d_n$

$$(\lambda - d_1)(\lambda - d_2)\cdots(\lambda - d_n) = \lambda^n + C_{n-1}\lambda^{n-1} + \cdots + C_0$$

characteristic polynomial does not change for a linear transformation T w.r.t. different basis.

Goal: Find a good basis $\{\vec{v}_1, \cdots, \vec{v}_n\}$. P invertible.

s.t. $P^{-1}\cdot A \cdot P = D \Leftrightarrow \cancel{P}\cancel{P^{-1}}AP = P\cdot D \Leftrightarrow A\cancel{P}\cdot\cancel{P^{-1}} = P\cdot DP^{-1}$

Algorithm: ④. $\det(A - \lambda I) = 0$ solve for λ_i

⑤ For each λ_i solve $(A - \lambda_i I)\cdot\vec{x} = \vec{0}$

$V_\lambda : \{\vec{v} \in \mathbb{R}^n \mid (A - \lambda_i I)\vec{v} = \vec{0}\}$ eigen space.

⑥ If $\sum_i \dim(V_{\lambda_i}) = n$ $\{\vec{v}_1, \cdots, \vec{v}_n\}$ are the basis for $\mathbb{R}^n$. $P = \begin{pmatrix} \downarrow & \cdots & \downarrow \\ v_1 & & v_n \end{pmatrix}$

then $P^{-1}AP = D$ where D_{ii} : e.v. for $\vec{v}_i$.

diagonalization $\longleftrightarrow$ complete the square.

3) Symmetric Matrix: $\longleftrightarrow$ quadratic form. in n variables.

$A = A^T \in M_{n\times n}(\mathbb{R})$

Thm: If A is symmetric, then there exists P

s.t. $P^{-1}AP = D$ and P is orthogonal matrix

$P^T \cdot A P = D$ (i.e. column vectors are orthonormal basis).

$P \cdot P^T = P^T \cdot P = I$

$$\underbrace{P^T}_{}\cdot P = \begin{pmatrix} \longrightarrow \vec{v}_1 \\ \longrightarrow \\ \vdots \\ \longrightarrow \vec{v}_n \end{pmatrix} \begin{pmatrix} \downarrow & \downarrow & \cdots & \downarrow \\ v_1 & & & \vec{v}_n \end{pmatrix} = I$$

Algorithm: ⑦ Classify types of quadratic forms.

sign of λ_i.

⑧ P orthogonal s.t.

$P^{-1}AP = D$

Gram-Schmidt. projection: of $\vec{v}$ along $\vec{u}$.

$$\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\cdot\vec{u}$$

Ex: If a linear transformation $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$.

$\lambda_1 = 1 \quad \lambda_2 = 1 \quad \lambda_3 = -1$

and $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

what is the matrix for T understandard basis?

$P^{-1}AP = D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \qquad P = \begin{pmatrix} v_1 & v_2 & v_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$

$A = PDP^{-1} \qquad P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$$P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

$$\begin{array}{cccccc} 0 & 1 & \cancel{-1} & \cancel{-1} & \cancel{0} & \cancel{1} \\ & 0 & \cancel{-2} & \cancel{-1} & \cancel{-1} & \cancel{1} \\ & & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

$$A = P \cdot D \cdot P^{-1} = \ldots$$