Week 3, Tuesday

1.3.

Recall $\vec{v}_1, \cdots, \vec{v}_m \in \mathbb{R}^n$, then. span of $\{\vec{v}_i\}$ is the set of all vectors in the format of $\sum \alpha_i \vec{v}_i$ (where $\alpha_i \in \mathbb{R}$)

Ex: $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} -3 \\ 1 \\ 8 \end{pmatrix} \in \mathbb{R}^3$

$\vec{y} = \begin{pmatrix} h \\ -5 \\ -3 \end{pmatrix}$, which values of h is $\vec{y}$ in the span of $\vec{v}_1$ and $\vec{v}_2$?

Ans: If x, y are two nubers s.t.

$\vec{y} = x \cdot \vec{v}_1 + y \cdot \vec{v}_2$ then

$$x \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + y \cdot \begin{pmatrix} -3 \\ 1 \\ 8 \end{pmatrix} = \begin{pmatrix} h \\ -5 \\ -3 \end{pmatrix} \Leftrightarrow$$

$$\begin{cases} x - 3y = h \\ x \cdot 0 + y = -5 \\ -2x + 8y = -3 \end{cases}$$ a.m. is $$\left(\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{array}\right) \begin{array}{c} ① \\ ② \\ ③ \end{array}.$$

To get an echelon form.

③ $\rightsquigarrow$ ③ + ① × 2

$$\left(\begin{array}{cc|c} \boxed{1} & -3 & h \\ 0 & \boxed{1} & -5 \\ 0 & 2 & -3+2h \end{array}\right)$$

③ $\rightsquigarrow$ ③ − ② × 2

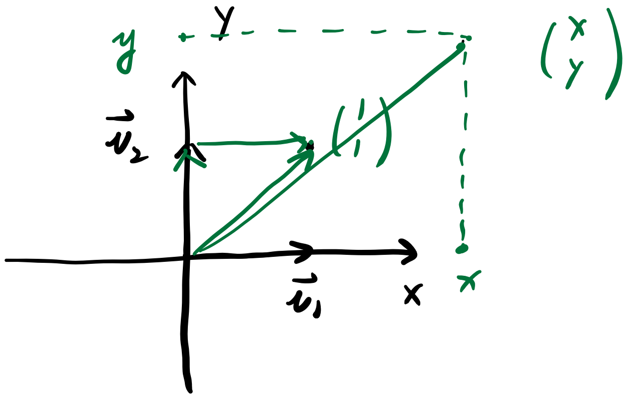
$$\left(\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 2h-3+10 \end{array}\right)$$

This echelon form has no pivot in the last column if and only if the linear system has a solution.

So $\vec{y}$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$, $\iff$ $2h+7=0$, i.e. $h=-\frac{7}{2}$.

Geometric picture of span.

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$x \cdot \vec{v}_1 + y \cdot \vec{v}_2 = \begin{pmatrix} x \\ y \end{pmatrix}$

We say that $\vec{v}_1$ and $\vec{v}_2$ span $\mathbb{R}^2$, or span $x-y$ plane.

Q: Does $\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\vec{u}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ span $\mathbb{R}^2$?

For every vector in $\mathbb{R}^2$, say $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ we can find

x_1, x_2 s.t. $x_1 \cdot \vec{u}_1 + x_2 \cdot \vec{u}_2 = \vec{y}$ because.

the a.m. $\left(\begin{array}{cc|c} 1 & 2 & y_1 \\ 2 & -3 & y_2 \end{array}\right)$ has an echelon form.

$$\left(\begin{array}{cc|c} \boxed{1} & 2 & y_1 \\ 0 & \boxed{-7} & y_2 - y_1 \times 2 \end{array}\right).$$

$\square$.

1.4 Matrix equation

Def. A. a $m \times n$ matrix, with column vectors $\vec{a}_1, \vec{a}_2, \cdots \vec{a}_n$, and $\vec{x} \in \mathbb{R}^n$. We define

$$A \cdot \vec{x} \triangleq (\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \cdot \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n.$$

Then linear system

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases} \iff x_1\cdot\vec{a_1} + x_2\cdot\vec{a_2} + \cdots + x_n\vec{a_n} = \vec{b}$$

$$\Updownarrow \text{ by def above}$$

$$A\cdot\vec{x} = \vec{b}$$

Therefore. the linear system has a solution if and only if. $\vec{b}$ is in the span of colum vectors in A, the c.m.

1.5 Solution set of linear systems.

Ex. Solve. both linear systems.

$$\begin{cases} x_1 + 2x_2 - x_3 = 0 \\ \qquad\; x_2 + x_3 = 0 \\ -x_1 + x_2 + 4x_3 = 0 \end{cases} \qquad \begin{cases} x_1 + 2x_2 - x_3 = 1 \\ \qquad\; x_2 + x_3 = 2 \\ -x_1 + x_2 + 4x_3 = 5 \end{cases}$$

a.m.

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ -1 & 1 & 4 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 1 \\ 0 & \boxed{1} & 1 & 2 \\ -1 & 1 & 4 & 5 \end{array}\right)$$

③ ⇝ ③ + ①. ③ ⇝ ③ + ①.

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 1 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 3 & 3 & 6 \end{array}\right)$$

③ ⇝ ③ - ② × 3 ③ ⇝ ③ - ② × 3 ⇝ $x_1 + 2x_2 - x_3 = 1$

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 1 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

basic variables. free variable

basic variables. free

$$\begin{cases} x_1 = x_3 - 2x_2 = 3x_3 \\ x_2 = -x_3 \\ x_3 \text{ is free} \end{cases} \qquad \begin{cases} x_1 = 1 + x_3 - 2\cdot x_2 = -3 + 3x_3 \\ x_2 = 2 - x_3 \\ x_3 \text{ is free} \end{cases}$$

$$\vec{x} = \begin{pmatrix} 3x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

parametric vector form

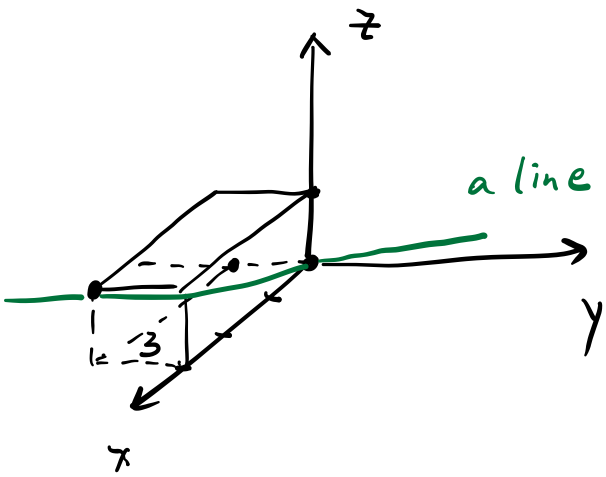
$$\vec{x} = \begin{pmatrix} -3+3x_3 \\ 2-x_3 \\ 0+x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

particular solution $A\vec{x} = \vec{b}$

general solution of $A\vec{x} = \vec{0}$

A: c.m.

The solution set is span of $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$



Notice that

$$\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

will also be a parametric vector form of the solution set.

Def. $A \cdot \vec{x} = \vec{0}$ is called homogeneous. linear system.

$A \cdot \vec{x} = \vec{b}$ with non-zero $\vec{b}$ is called non-homogeneous.

Thm. Suppose $A\vec{x} = \vec{b}$ has a solution $\vec{p}$. Then the set of solutions to $A\vec{x} = \vec{b}$ contains exactly vectors of the form $\vec{p} + \vec{v}$ where $\vec{v}$ is a solution to $A\vec{x} = \vec{0}$

pf. ① Firstly check that all vectors $\vec{p} + \vec{v}$ are solutions of $A\vec{x} = \vec{b}$.

$$A \cdot (\vec{p} + \vec{v}) = A \cdot \vec{p} + A \cdot \vec{v} = \vec{b} + \vec{0} = \vec{b}.$$

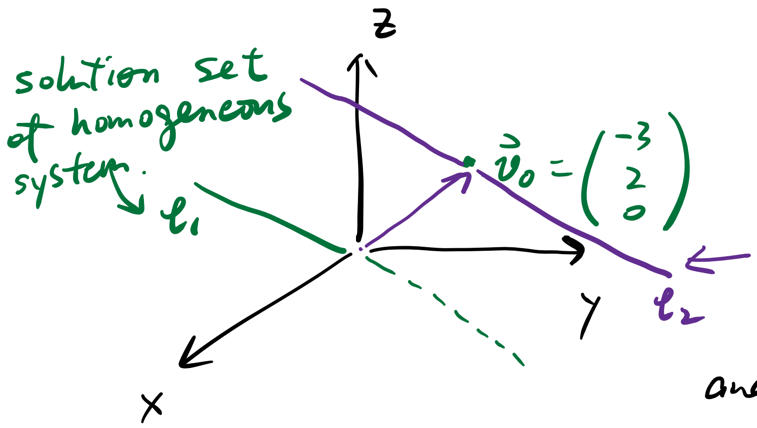
② Secondly, check that all solutions can be written in this format.

Suppose $\vec{u}$ is a solution. i.e. $A\cdot\vec{u}=\vec{b}$.

then $\vec{u}=\vec{p}+(\vec{u}-\vec{p})$ and then.

$$A\cdot(\vec{u}-\vec{p}) = A\cdot\vec{u} - A\cdot\vec{p} = \vec{b} - \vec{b} = \vec{0}. \qquad \square.$$

Geometric Picture.



solution set of non-homogeneous. unique line that is passing $\vec{v}_0$ and parallel to ℓ_1

Ex. $\begin{cases} 2x_1 + x_2 - x_3 + 2x_4 = 3 \\ 4x_1 - x_2 - 2x_3 + 7x_4 = 3 \end{cases}$

Solve this system and write solution into parametric vector form.

a.m. $\left(\begin{array}{cccc|c} \boxed{2} & 1 & -1 & 2 & 3 \\ 4 & -1 & -2 & 7 & 3 \end{array}\right)$

② ⟶ ② − ① ×2

$\left(\begin{array}{cccc|c} \boxed{2} & 1 & -1 & 2 & 3 \\ 0 & \boxed{-3} & 0 & 3 & -3 \end{array}\right)$

basic variables. free varibables

$$\begin{cases} x_1 = (3 - 2x_4 + x_3 - x_2)/2 = 1 + \frac{x_3}{2} - \frac{3}{2}x_4 \\ x_2 = 1 + x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases}$$

So the solution is

$$\vec{x} = \begin{pmatrix} 1 + \frac{x_3}{2} - \frac{3}{2}x_4 \\ 1 + x_4 \\ x_3 \\ x_4 \end{pmatrix} = \underset{\substack{\text{particular} \\ \text{solution of} \\ A\vec{x}=\vec{b}.}}{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}} + \underbrace{x_3 \cdot \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} -\frac{3}{2} \\ 1 \\ 0 \\ 1 \end{pmatrix}}_{\substack{\text{general solution to} \\ A\vec{x}=0}}$$

Conclusion: If $A\vec{x} = \vec{b}$ has a unique solution, then $A\vec{x} = \vec{0}$ also has a unique solution.

Q: Does the converse holds?