Week 4 Tuesday

1.7 Linear indepence

Def. $\vec{v}_1, \cdots, \vec{v}_p \in \mathbb{R}^n$ are linearly independent if

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_p \vec{v}_p = \vec{0}$$

has only trivial solution ($x_1 = x_2 = \cdots = x_p = 0$).

*: when $\vec{v}_1, \cdots, \vec{v}_p$ has non-trivial solution we say they are linearly dependent.

Recall that vector equation is equivalent to a linear system. In particular, this linear system is a homogeneous system where c.m. has n rows and p columns.

Conclusion: The homogeneous system $A\vec{x} = \vec{0}$ has only trivial solution $\vec{x} = \vec{0}$ $\Leftrightarrow$ Column vectors of A are linearly independent.

EX. $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \in \mathbb{R}^2$

$$x_1 \cdot \vec{v}_1 + x_2 \cdot \vec{v}_2 = \vec{0} \Leftrightarrow \begin{cases} x_1 + 2x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{cases}$$

a.m. is $\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 6 & 0 \end{array}\right)$ ② $\rightsquigarrow$ ② $-$ 3 × ①. $\left(\begin{array}{cc|c} \boxed{1} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

free variable

∞ solutions $\Rightarrow$ $\vec{v}_1$ and $\vec{v}_2$ are not linearly independent.

Or. another way is to observe that.

$2 \cdot \vec{v}_1 - 1 \cdot \vec{v}_2 = \vec{0}$ so $x_1 = 2$ $x_2 = 1$ is a non-trivial solution.

Ex. $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

a.m. $\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 5 & 0 \end{array}\right)$ ② ⇝ ② − 3 × ① $\left(\begin{array}{cc|c} \boxed{1} & 2 & 0 \\ 0 & \boxed{-1} & 0 \end{array}\right)$

so unique solution $\vec{x} = \vec{0}$.

Geometric Picture.

Thm. If $\vec{v}_1, \dots, \vec{v}_p$ are linearly dependent, then. $\exists\, j$ s.t.

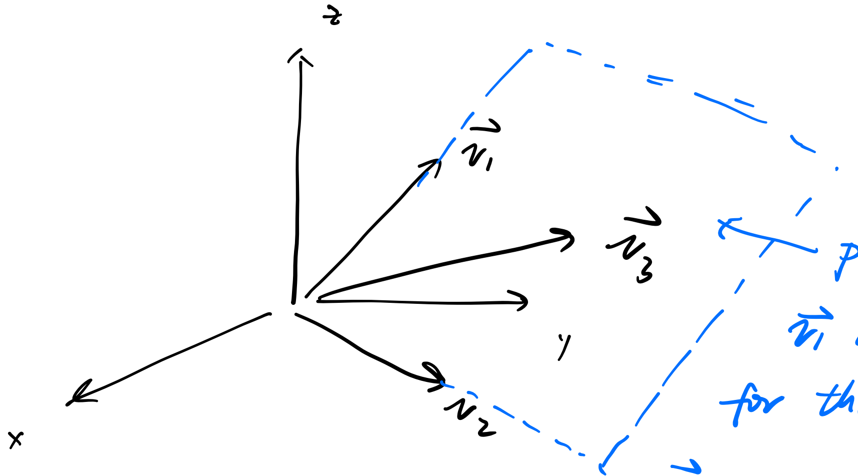
$\vec{v}_j$ is a linear combination of $\{\vec{v}_i \mid i \neq j\}$.

Pf. There exists $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \neq \vec{0}$ s.t.

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_p \cdot \vec{v}_p = 0$$

so. if $x_j \neq 0$ then $\vec{v}_j = -\frac{1}{x_j} \cdot (\underbrace{x_1 \vec{v}_1 + \vec{x}_2 \vec{v}_2 + \dots + x_p \vec{v}_p}_{p-1 \text{ term, no } j\text{-term}})$.

$\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$



plane containing span of $\vec{v}_1$ and $\vec{v}_2$. the equation for the plane is.

$\vec{x}(u,v) = \vec{v}_1 \cdot u + \vec{v}_2 \cdot v$

Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent then $\vec{v}_3$ is on the plane spanned by $\vec{v}_1$ and $\vec{v}_2$.

$\vec{x}(t) = \vec{v}_0 + t \cdot \vec{v}_1$

$\vec{x}(u,v) = \vec{v}_0 + u \cdot \vec{v}_1 + v \cdot \vec{v}_2$

In another words, $\vec{v}_3$ is not necessary to span this plane, or. $\text{span}\{\vec{v}_1, \vec{v}_2\} = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

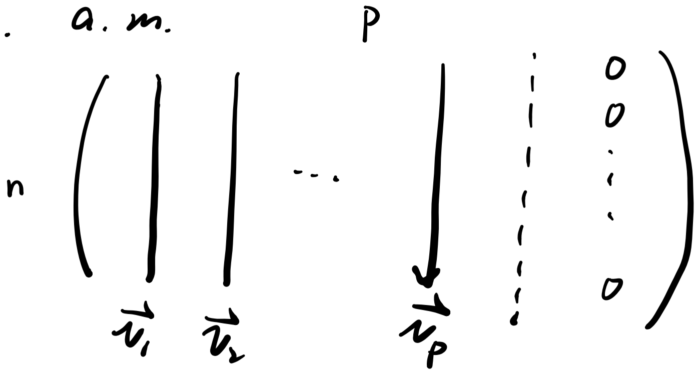
So given $\vec{v}_1, \cdots, \vec{v}_p$, if we want to build a linearly independent set that spans $\text{span}\{\vec{v}_1, \cdots, \vec{v}_p\}$, we just need to delete $\vec{v}_j$ that can be written as a linear combination of $\vec{v}_1, \cdots, \vec{v}_{j-1}, \vec{v}_{j+1}, \cdots, \vec{v}_p$.

Thm. Let $\vec{v}_1, \cdots, \vec{v}_p \in \mathbb{R}^n$, If $p > n$, then $\vec{v}_1, \cdots, \vec{v}_p$ are linearly dependent.

Pf. The linearly system corresponding to

$$x_1 \vec{v}_1 + \cdots + x_p \cdot \vec{v}_p = \vec{0}$$

has. a. m.



Since $n < p$, we cannot find pivot for every column. so there must be free variable. so there are ∞

solutions. So $\vec{v}_1, \cdots, \vec{v}_p$ are linearly dependent.

Ex. $\vec{v}_1 = \begin{pmatrix}1\\0\\2\\3\end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix}2\\1\\-1\\2\end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix}3\\1\\1\\-1\end{pmatrix}$.

Are they lin. dep ?

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 2 & -1 & 1 & 0 \\ 3 & 2 & -1 & 0 \end{array}\right)$$

③ $\rightsquigarrow$ ③ $- 2\times$ ①

④ $\rightsquigarrow$ ④ $- 3\times$ ①.

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & -5 & -5 & 0 \\ 0 & -4 & -10 & 0 \end{array}\right)$$

③ $\rightsquigarrow$ ③ $+ 5\times$ ②

④ $\rightsquigarrow$ ④ $+ 4\times$ ②

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right)$$

③ $\leftrightarrow$ ④

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{-6} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So there is one unique solution $x_i = 0$. So lin independent.

Ex. $\vec{v}_1 = \begin{pmatrix}1\\0\\2\\3\end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix}2\\1\\-1\\2\end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix}3\\1\\1\\1\end{pmatrix}$ $\vec{v}_4 = \begin{pmatrix}-1\\2\\3\\5\end{pmatrix}$

$\vec{v}_5 = \begin{pmatrix}3\\-1\\-2\\-4\end{pmatrix}$. lin dep since $p > n$.

1.8 Linear transformations

A fuction $f: \mathbb{R} \longrightarrow \mathbb{R}$ is a map that specifies one output for each input.

e.g. $f(x) = \underline{\underline{ax+b}}$. $f(x) = x^2$, $\dots$, $f(x) = \sin x$. $f(x) = \arcsin x$. $\in [0, 2\pi)$

$ax+b$ is a linear fution. it looks like a line.

We can generalize. $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

eg. linear combinations of vector.

given $\vec{v}_0$, $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^m$. then.

$$f: \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \rightarrow \vec{y} = x_1 \cdot \vec{v}_1 + \cdots + x_n \cdot \vec{v}_n \in \mathbb{R}^m$$
$$+ \vec{v}_0$$

$$\vec{y} = A \cdot \vec{x} + \vec{v}_0$$

Linear Transformation:

A fution f from $\mathbb{R}^n$ to $\mathbb{R}^m$ is called <u>linear</u>

if

1) $f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$

2) $f(c \cdot \vec{u}) = c \cdot f(\vec{u})$. $\forall c \in \mathbb{R}$

eg. If $f(\vec{x}) = A \cdot \vec{x}$ then.

$$f(\vec{u} + \vec{v}) = A \cdot (\vec{u} + \vec{v}) = A \cdot \vec{u} + A \cdot \vec{v} = f(\vec{u}) + f(\vec{v})$$

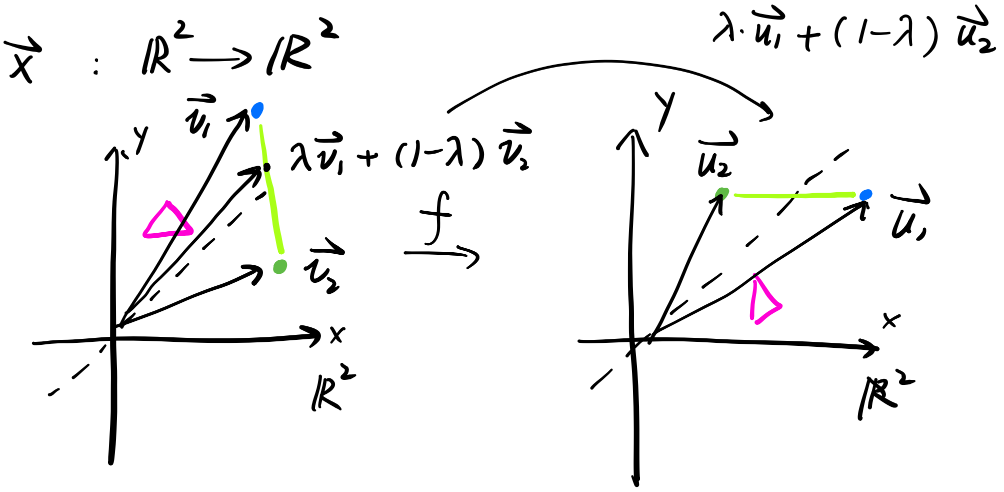
$$f(c \cdot \vec{u}) = A \cdot (c\vec{u}) = c \cdot (A \cdot \vec{u}) = c \cdot f(\vec{u}).$$

Several Class of linear transformations:

in $\mathbb{R}^2$ we can give A in different taste:

- $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\vec{y} = A\vec{x}$: $\mathbb{R}^2 \to \mathbb{R}^2$

  $(x, y) \to (y, x)$



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