Week 3 Thursday.

Next week Sept 16 room change

Recall $\vec{y} = A\cdot\vec{x} + \vec{b}$ $\quad \mathbb{R}^m \longrightarrow \mathbb{R}^n$

<u>Linear Transformation</u>: $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$

- $T(\vec{x}_1 + \vec{x}_2) = T(\vec{x}_1) + T(\vec{x}_2)$
- $T(c\cdot\vec{x}) = c\cdot T(\vec{x})$

Claim: $T(\vec{0}) = \vec{0}$

$T(\vec{0}+\vec{0}) = T(\vec{0})$

$\parallel$

$T(\vec{0}) + T(\vec{0}) \Rightarrow T(\vec{0}) = \vec{0}$.

Let $T(\vec{x}) = A\vec{x} + \vec{b}$

$T(\vec{x}_1 + \vec{x}_2) = A\cdot(\vec{x}_1 + \vec{x}_2) + \vec{b}$

$= A\vec{x}_1 + A\vec{x}_2 + \vec{b} \overset{?}{=} T(\vec{x}_1) + T(\vec{x}_2)$

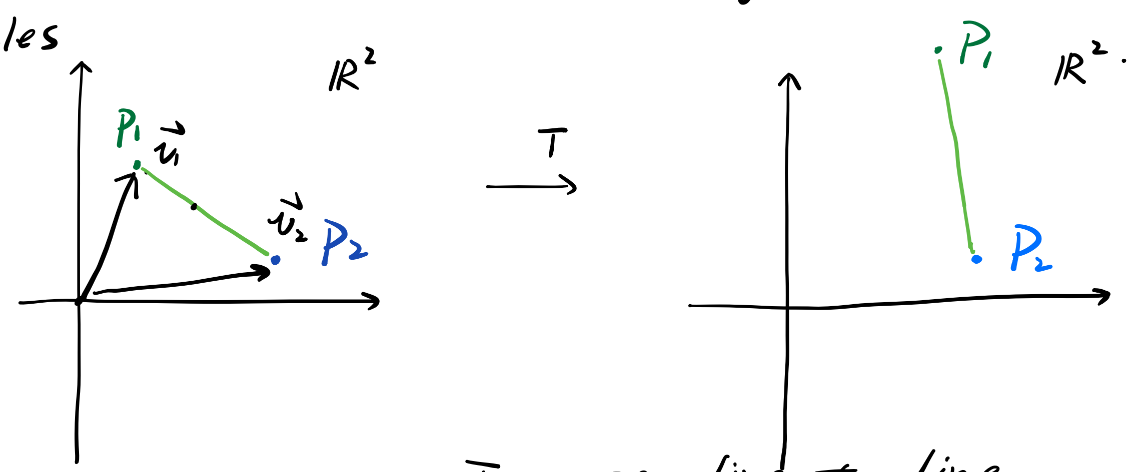
The equality only holds when $\vec{b} = \vec{0}$.

If $\vec{b} = \vec{0}$, then. $T(c\vec{x}) = A\cdot(c\vec{x}) = c\cdot(A\vec{x})$

$= c\cdot T(\vec{x})$

Conclusion: $\vec{y} = A\vec{x}$ are always linear tranformation.

Examples



Linear Transformation T maps line to line.

Pf: Points on the line connecting P_1 and P_2 can be written as $\vec{v} = \lambda\cdot\vec{v}_1 + (1-\lambda)\vec{v}_2$ for $0 \le \lambda \le 1$ (little ex for you)

By linearality of T.

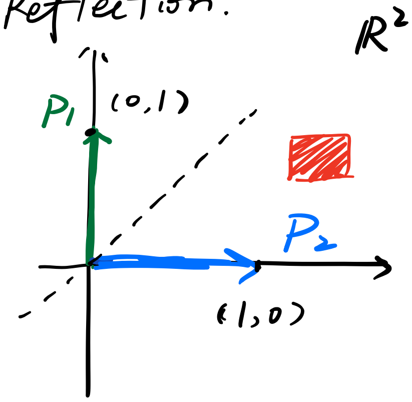
$T(\vec{v}) = T(\lambda\vec{v}_1 + (1-\lambda)\vec{v}_2) = T(\lambda\vec{v}_1) + T((1-\lambda)\vec{v}_2)$

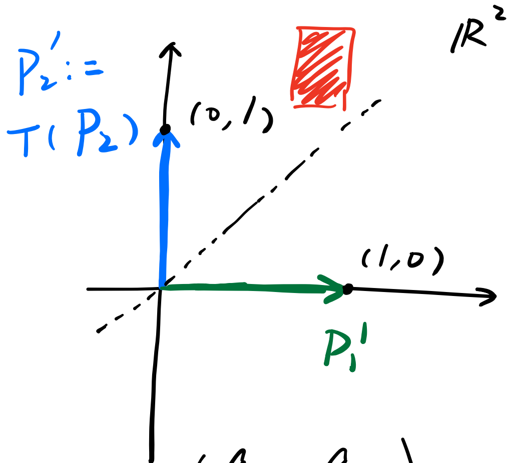
$= \lambda T(\vec{v}_1) + (1-\lambda) T(\vec{v}_2)$

which is another point connecting P_1 and P_2 on the right hand side.

Concrete Example:

· Reflection.





Suppose A is this matrix. $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then.

$$A(\vec{x}) = x_1 \cdot \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

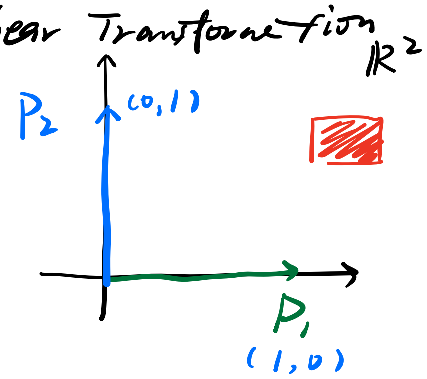
$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\Rightarrow$ $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ gives this

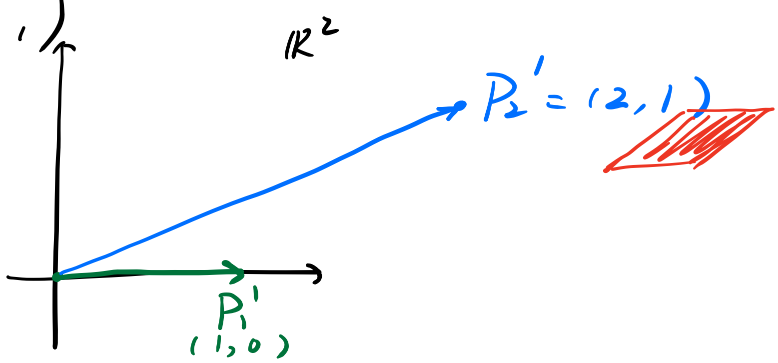
$$A \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

l.t.

· Shear Transformation $\qquad A = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

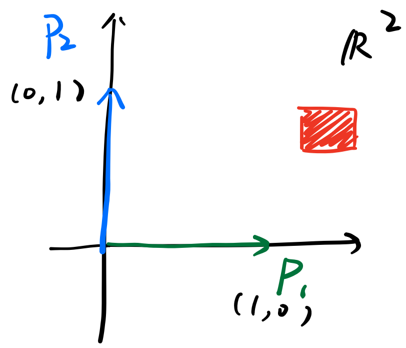


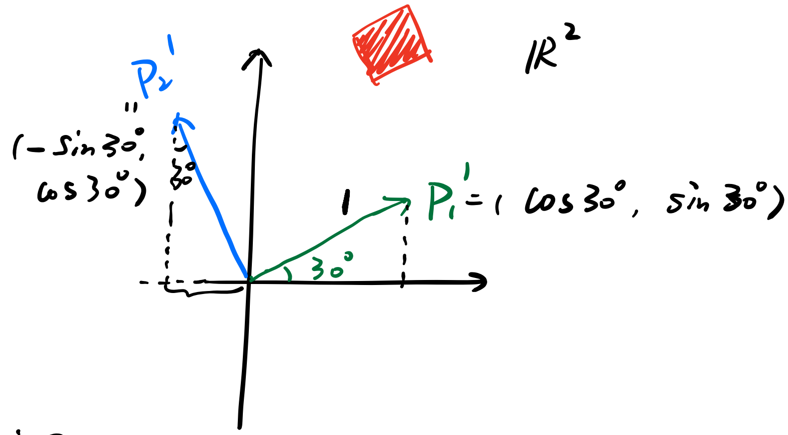
T



$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

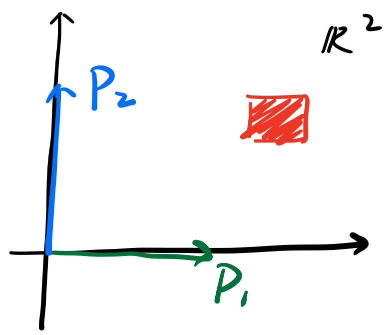
- Rotation.

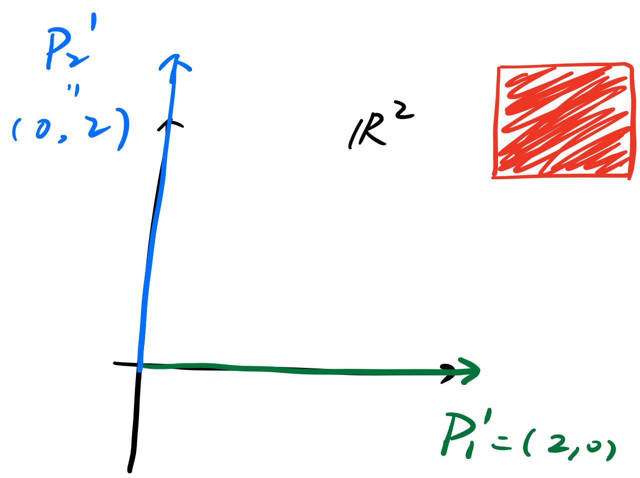




$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

- Dilation / Contraction. $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ $\lambda > 0$.

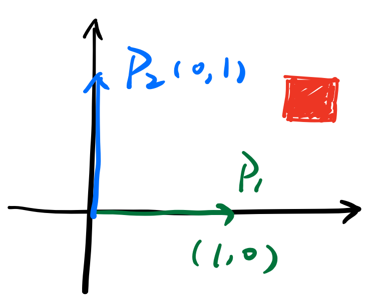


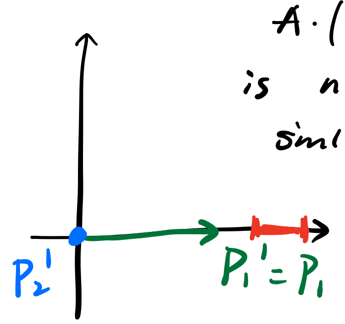


$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

- Projection. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$T = A \cdot \vec{x}$



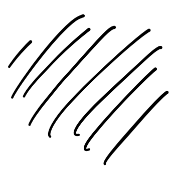


is not injective.

$A \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = A \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{0}$

is not surjective.

since $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is not in the image.



$\mathbb{R}^2$

domain

image

x-axis in $\mathbb{R}^2$

$T: \mathbb{R}^m \to \mathbb{R}^n$.

Def. A linear transformation is *injective* if for each $\vec{y} \in \mathbb{R}^n$, there exists at most one $\vec{x} \in \mathbb{R}^m$ s.t. $T(\vec{x}) = \vec{y}$. T is *surjective* (or *onto*) if for every $\vec{y} \in \mathbb{R}^n$. there exists at least one $\vec{x} \in \mathbb{R}^m$ s.t $T(\vec{x}) = \vec{y}$.

Thm. 1) Any linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ can be described by $T(\vec{x}) = A \cdot \vec{x}$.

2). T is injective iff $\vec{0}$ is the only preimage of $\vec{0}$.

Pf. 1) Denote $\vec{e}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ i-th row. $1 \le i \le m$

and denote. $\vec{d}_i = T(\vec{e}_i)$ then.

$$T(x_1 \vec{e}_1 + x_2 \vec{e}_2 + \cdots x_m \cdot \vec{e}_m) = x_1 \cdot T(\vec{e}_1) + \cdots + x_m T(\vec{e}_m)$$

$\quad\;\; \parallel$

$T(\vec{x})$ for $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$

$$= x_1 \cdot \vec{d}_1 + \cdots + x_m \cdot \vec{d}_m.$$

$$= \begin{array}{c} m \\ n \left(\begin{array}{cccc} \vec{d}_1 \downarrow & \downarrow & \cdots & \vec{d}_m \downarrow \end{array} \right) \end{array} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}.$$

So $T = A \cdot \vec{x}$ where A has colum vector $\vec{d}_j$. $1 \le j \le m$.

2). $\Rightarrow$ Trivial. by def of injective.

$\Leftarrow$ Suppose T is not injective. and $\exists$ $\vec{x}_1 \ne \vec{x}_2$ s.t.

$$T(\vec{x}_1) = T(\vec{x}_2)$$

then $T(\vec{x}_1 - \vec{x}_2) = T(\vec{x}_1) - T(\vec{x}_2) = \vec{0}$

So. $\vec{x}_1 - \vec{x}_2 \ne 0$ is another preimage of $\vec{0}$. $\square$.