Week 4 Tuesday.

1. Thursday

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2. Next Tuesday.

1st mid-term.

1h 15 min

Range : matrix inverse.

No Calculator.

Recall linear transformation

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

1) $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$

2) $T(c \cdot \vec{v}) = c \cdot T(\vec{v})$

T is always in the format of a matrix. examples of T:

rotation, dilation, reflection, projection, shear transformation.

Q1:

$$\mathbb{R}^n \xrightarrow{B} \mathbb{R}^m \xrightarrow{A} \mathbb{R}^r$$

Is the composition still linear?

Q2: Can you build up all linear $T: \mathbb{R}^2 \to \mathbb{R}^2$ via taking compositions of the examples listed above?

For Q1 :

Prop. Given linear transformations A & B. $\mathbb{R}^n \xrightarrow{B} \mathbb{R}^m \xrightarrow{A} \mathbb{R}^r$ their composition. $C = A \circ B : \mathbb{R}^n \to \mathbb{R}^r$. is still linear.

proof:

$$\begin{aligned}
(A \circ B)(\vec{v}_1 + \vec{v}_2) &\overset{\text{def of comp}}{=} A\left(B(\vec{v}_1 + \vec{v}_2)\right) \\
&\overset{B \text{ is linear}}{=} A\left(B(\vec{v}_1) + B(\vec{v}_2)\right) \\
&\overset{A \text{ is linear}}{=} A(B(\vec{v}_1)) + A(B(\vec{v}_2)) \\
&\overset{\text{def of comp}}{=} (A \circ B)(\vec{v}_1) + (A \circ B)(\vec{v}_2)
\end{aligned}$$

$$(A \circ B)(c \cdot \vec{v}) = A(B(c\vec{v})) = A(c\,B(\vec{v})) = c \cdot A(B(\vec{v}))$$

$$= c \cdot (A \circ B)(\vec{v})$$

$\square$.

Q3: what is the matrix representing $A \circ B$?

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

then $\quad \mathbb{R}^2 \xrightarrow{B} \mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2$

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{B} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = b_{11}\cdot\vec{e}_1 + b_{21}\cdot\vec{e}_2 \xrightarrow{A} b_{11}\cdot A(\vec{e}_1) + b_{21}\cdot A(\vec{e}_2)$$

$$= b_{11}\cdot\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{21}\cdot\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

$$\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{B} \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix} = b_{12}\cdot\vec{e}_1 + b_{22}\cdot\vec{e}_2$$

$$\downarrow A$$

$$= b_{12}\cdot\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{22}\cdot\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} & \\ & \end{pmatrix}$$

So the matrix for $C = A \circ B$ is

$$\begin{pmatrix} \underbrace{a_{11}\cdot b_{11}}_{k=1} + \underbrace{a_{12}\cdot b_{21}}_{k=2} & \underbrace{a_{11}\cdot b_{12}}_{k=1} + \underbrace{a_{12}\cdot b_{22}}_{k=2} \\ a_{21}\cdot b_{11} + a_{22}\cdot b_{21} & a_{21}\cdot b_{12} + a_{22}\cdot b_{22} \end{pmatrix}$$

Def: Given $B \in M_{m\times n}$, $A \in M_{r\times m}$ we define

$A\cdot B$ by $\quad (A\cdot B)_{ij} = \sum_{k=1}^{m} A_{ik}\cdot B_{kj}$.

We define matrix multiplication to catch the matrix for compositions.

Operations for Matrix (Matrix Algebra)

$A \in M_{m\times n}$ $B \in M_{m\times n}$

$\pm$: $(A \pm B)_{ij} = A_{ij} \pm B_{ij}$

scalar multiplication : $(k \cdot A)_{ij} = k \cdot A_{ij}$

If. $A \in M_{m\times n}$ $B \in M_{n\times r}$ then

matrix multiplication: $(A \cdot B)_{ij} = \sum_k A_{ik} \cdot B_{kj}$

For square matrix, we can define all of them for any two matrix $A, B \in M_{n\times n}$.

<u>Property of Matrix Multiplication</u> :

1) $A \cdot (B \cdot C) = (A \cdot B) \cdot C$

2) $A \cdot (B + C) = A \cdot B + A \cdot C$

3) $(B + C) \cdot A = BA + C \cdot A$.

4) $r \cdot (A \cdot B) = (rA) \cdot B = A \cdot (rB)$

5) $I_m \cdot A = A = A \cdot I_n$

pf. 1).
$$[A \cdot (B \cdot C)]_{i,j} = \sum_k A_{ik} \cdot (B \cdot C)_{kj}$$
$$= \sum_k A_{ik} \cdot \sum_\ell B_{k\ell} \cdot C_{\ell j}$$
$$= \sum_{k,\ell} A_{ik} \cdot B_{k\ell} \cdot C_{\ell j}$$
$$[(A \cdot B) \cdot C]_{i,j} = \sum_\ell (A \cdot B)_{i\ell} \cdot C_{\ell j}$$
$$= \sum_\ell \sum_k A_{ik} \cdot B_{k\ell} \cdot C_{\ell j}$$

2) 3) 4). exercise $\square$.

Rmk: ① $I_m \cdot A = A$. $\quad r \cdot A = \begin{pmatrix} r & & & \\ & r & & \\ & & \ddots & \\ & & & r \end{pmatrix} \cdot A$.

② Check that

$A \cdot \vec{x}$ (in the sense of matrix vector multiplication)

equals $A \cdot \vec{x}$ (in the sense of matrix multiplication)

③. $AB \neq BA$

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1\times 1 + a\times b & 1\times 0 + a\times 1 \\ b & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+ab & a \\ b & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ b & 1+ab \end{pmatrix}$$

④ $AB = AC \overset{\times}{\Longrightarrow} B = C$

$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$

$A \cdot (B - C) = 0 \qquad B - C = 0$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{21} & b_{22} \\ 0 & 0 \end{pmatrix}$$

so $\underset{\uparrow \atop A}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \underset{\uparrow \atop B}{\begin{pmatrix} c & d \\ a & b \end{pmatrix}} = \underset{\uparrow \atop A}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \underset{\uparrow \atop C.}{\begin{pmatrix} e & f \\ a & b \end{pmatrix}}$

$B \neq C$ in general.

⑤. $A \cdot B = 0 \overset{\times}{\Longrightarrow} A = 0$ or $B = 0$.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$