Week 4 Thursday.

Recall from last time:

$$(A\cdot B)_{ij} = \sum_k A_{ik}\cdot B_{kj} = \begin{pmatrix} A_{i1} \\ A_{i2} \\ \vdots \\ A_{in} \end{pmatrix} \cdot \begin{pmatrix} B_{1j} \\ B_{2j} \\ \vdots \\ B_{nj} \end{pmatrix}$$

$A\cdot B = A\cdot C$ then $\overset{\times}{\Rightarrow} B=C$

if $a\cdot b = a\cdot c$ for $a,b,c \in \mathbb{R}$. then. again we do not know $b=c$, but if $a\neq 0$ then multiply by $\frac{1}{a} = a^{-1}$ on both sides. $\Rightarrow b=c$.

For matrix, we also give a analogons condition for cancellation to hold, that is, invertible.

Def. Given $A\in M_{n\times n}$. if. $A\cdot B = B\cdot A = I$. then. B is called A^{-1} and we say A is invertible.

Q1: Is inverse unique?

If B_1, B_2 are both inverses of A.

$B_1\cdot A = A\cdot B_1 = I$

$B_2\cdot A = A\cdot B_2 = I$

$A\cdot(B_1-B_2) = 0 \Rightarrow B_1\cdot A\cdot(B_1-B_2) = 0 \Rightarrow B_1-B_2=0$

Property

① Yes! The inverse is unique.

②: If A and B are both invertible. then.

$A \cdot B$ is also invertible.

$$\left.\begin{array}{l} (A\cdot B)\cdot(B^{-1}\cdot A^{-1}) = I \\ (B^{-1}\cdot A^{-1})(A\cdot B) = I. \end{array}\right\} \Rightarrow \quad (A\cdot B)^{-1} = B^{-1}\cdot A^{-1}.$$

③ $(A^{-1})^{-1} = A$

$$A\cdot A^{-1} = I$$
$$A^{-1}\cdot A = I.$$

Recall matrix multiplication gives the matrix for composition of linear transformations. say if

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is linear with standard matrix A.

then. A is invertible if and only if T is injective and surjective.

$\exists B$ s.t. $B\cdot A = A\cdot B = I$

$\Leftrightarrow \exists S: \mathbb{R}^n \to \mathbb{R}^n$ s.t $S\circ T = T\circ S = id.$

$\Leftrightarrow S = T^{-1}$ as a map.

So this provide us with a way to check whether A is invertible.

eg. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ To check A is injective or not. we solve the linear system.

$A \cdot \vec{x} = \vec{0}$ $\quad \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} \boxed{1} & 2 & 0 \\ 0 & \boxed{-2} & 0 \end{array}\right)$

no free variable, so A is invertible.

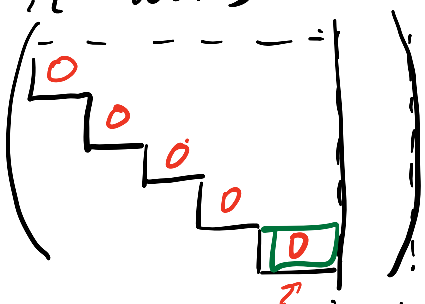
To check surjective: for any $\vec{y} \in \mathbb{R}^2$

$A \cdot \vec{x} = \vec{y}$ $\quad \left(\begin{array}{cc|c} 1 & 2 & y_1 \\ 3 & 4 & y_2 \end{array}\right) \to \left(\begin{array}{cc|c} \boxed{1} & 2 & * \\ 0 & \boxed{-2} & * \end{array}\right)$

Q: Is it possible to find $A: \mathbb{R}^n \to \mathbb{R}^n$ s.t A is injective but not surjective?

Thm. If $T: \mathbb{R}^n \to \mathbb{R}^n$ is linear then T is injective $\Leftrightarrow$ T is surjective.

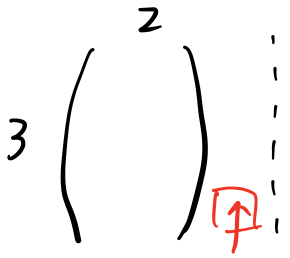
Pf: The matrix for T, call it by A. the echelon form for A has no free variables. implies that. it looks like.



So there are no pivot for the last column. This shows inj $\Rightarrow$ surj.

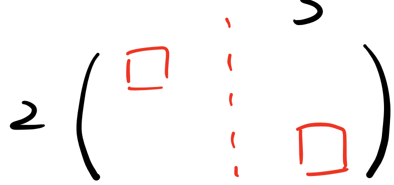
surj $\Rightarrow$ inj: In order for $A \cdot \vec{x} = \vec{y}$ to be consistent, we need the last column free of pivot. Since $\vec{y}$ is arbitrary, we must have $\boxed{}$ being a pivot. So the echelon form must look like above. $\square$.

$\mathbb{R}^2 \longrightarrow \mathbb{R}^3$



Cannot be surjectie

$\mathbb{R}^3 \longrightarrow \mathbb{R}^2$



cannot be injective.

In linear system perspective:

Application for A^{-1}:

$$A \cdot \vec{x} = \vec{b}$$

if A is invertible and. A^{-1} is the inverse.

then. multiply A^{-1} on both sides.

$$A^{-1} \cdot A \cdot \vec{x} = A^{-1} \cdot \vec{b}$$

$$\Rightarrow \quad \vec{x} = A^{-1} \cdot \vec{b}$$

Computation for A^{-1}:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix}$$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} = I.$$

$$\begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = I.$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \overset{?}{\cancel{=}} \begin{pmatrix} 1^{-1} & 1^{-1} \\ 1^{-1} & 1^{-1} \end{pmatrix}$$

$$\| $$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

One way suggested: $A: \underset{X}{\mathbb{R}^n} \longrightarrow \underset{Y}{\mathbb{R}^n}$, $\xleftarrow{A^{-1}}$

Determine the preimage of $\vec{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow i\text{th}$ under the map A. That is equivalent to solving $A \cdot \vec{x} = \vec{e}_i$.

Suppose the solution is $\vec{x}_i$, then

$$A^{-1} = \begin{pmatrix} \downarrow & \downarrow & \cdots & \downarrow \\ x_1 & x_2 & & x_n \end{pmatrix}$$

(Another way later we will see determinant).

Row operation for matrix is equivalent to certain matrix multiplication.

e.g. $A = \begin{pmatrix} \xrightarrow{r_1} \\ \xrightarrow{r_2} \\ \vdots \\ \xrightarrow{r_k} \end{pmatrix}$ $\qquad \tilde{A} = \begin{pmatrix} \xrightarrow{r_1} \\ \xrightarrow{r_2 + \lambda \cdot r_1} \\ \vdots \\ \xrightarrow{r_k} \end{pmatrix}$

$$= \begin{pmatrix} 1 & & & 0 \\ \lambda & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} \xrightarrow{r_1} \\ \xrightarrow{r_2} \\ \vdots \\ \xrightarrow{r_k} \end{pmatrix}$$

Similarly. $\tilde{A} = \begin{pmatrix} \vec{r}_2 \\ \vec{r}_1 \\ \vdots \\ \vec{r}_k \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & 0 \\ 1 & 0 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix} \begin{pmatrix} \xrightarrow{r_1} \\ \xrightarrow{r_2} \\ \vdots \\ \xrightarrow{r_k} \end{pmatrix}$

By a sequence of row operations (equivalently left multiplication by some invertible matrix), we get the echeloform of A. i.e. if

$$R_\ell \cdots R_2 R_1 A = I. \quad \text{then.}$$

$A \cdot \vec{x}_i = \vec{e}_i$ has the solution that.

$$\vec{x}_i = R_\ell \cdot R_{\ell-1} \cdots R_2 R_1 \cdot \vec{e}_i .$$

$$\left(\begin{array}{c|ccccc} A & 1 & 0 & 0 & & 0 \\ & 0 & 1 & 0 & & 0 \\ & 0 & 0 & 1 & \cdots & \vdots \\ & \vdots & \vdots & \vdots & & \vdots \\ & 0 & 0 & 0 & & 1 \end{array} \right)$$

I_n

Find row operations $R_\ell \cdot R_{\ell-1} \cdots R_2 R_1$ s.t.

$$A \xrightarrow{R_1} A_1 \xrightarrow{R_2} A_2 \rightarrow \cdots \quad \rightarrow I$$

then. Carry $R_\ell \cdot R_{\ell-1} \cdots R_2 R_1$ in the same way to I_n, we will get A^{-1} when LHS become I.

$$\underset{\overset{\shortparallel}{A}}{\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \right)} \xrightarrow{② \to ② - ① \times 3} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array} \right)$$

$$\xrightarrow{② \to ② / -2} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array} \right) \xrightarrow{① \to ① - ② \times 2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array} \right)$$

$$\begin{matrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{matrix} = A^{-1}$$