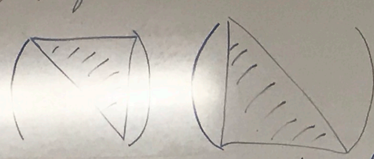
Recall upper/lower triangular matrix

has the format 

We show that if $A \in M_{n\times n}(\mathbb{R})$ can be reduced to echelon form without switching rows, then we can write $A = L\cdot U$

$$\underbrace{R_\ell \cdots R_2 R_1} A = U \longrightarrow A = \underbrace{R_1^{-1}\cdots R_{\ell-1}^{-1} R_\ell^{-1}}_{L} U$$

Suppose $A = L\cdot U$.

To solve

$$A\cdot\vec{x} = \vec{b} \Rightarrow L\cdot U\vec{x} = \vec{b} \Rightarrow U\vec{x} = L^{-1}\cdot\vec{b}$$

Similar idea in computing A^{-1}: instead of solving $A\vec{x} = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$ and so on.

$$\left(\begin{array}{c|cccc} A & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \cdots & 0 \\ & \vdots & 0 & & \vdots \\ & 0 & 0 & \cdots & 1 \end{array}\right)$$

eg. $A = \begin{pmatrix}\boxed{1} & 2 & 1\\ 1 & 0 & 0\\ 0 & 1 & 1\end{pmatrix} \xrightarrow[R_1 = \begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}]{② \rightsquigarrow ② - ①} R_1 A = \begin{pmatrix}\boxed{1} & 2 & 1\\ 0 & \boxed{-2} & -1\\ 0 & 1 & 1\end{pmatrix} \xrightarrow[R_2 = \begin{pmatrix}1&&\\0&1&\\0&\frac{1}{2}&1\end{pmatrix}]{③ \rightsquigarrow ③ + ② \times \frac{1}{2}} R_2 R_1 A = \begin{pmatrix}1 & 2 & 1\\ 0 & -2 & -1\\ 0 & 0 & \frac{1}{2}\end{pmatrix} = U.$

$$L = R_1^{-1}\cdot R_2^{-1} = \begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&-\frac{1}{2}&1\end{pmatrix}$$

$R_2 \cdot R_1 \cdot A = U$

$R_2^{-1} \underbrace{R_2 R_1 \cdot A} = R_2^{-1} U$

$A = R_1^{-1}\cdot R_2^{-1}\cdot U$

$$= \begin{pmatrix}1&0&0\\1&1&0\\0&-\frac{1}{2}&1\end{pmatrix}$$

$A = L\cdot U$

Suppose we solve $A\cdot\vec{x} = \begin{pmatrix}3\\0\\2\end{pmatrix} = \vec{b}$.

$\Leftrightarrow$ solve $U\cdot x = L^{-1}\cdot\vec{b}$

$$L^{-1} = R_2\cdot R_1 = \begin{pmatrix}1&0&0\\-1&1&0\\-\frac{1}{2}&1&1\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0\\-1&1&0\\-\frac{1}{2}&1&1\end{pmatrix}\begin{pmatrix}3\\0\\2\end{pmatrix} = \begin{pmatrix}3\\-3\\1/2\end{pmatrix}$$

$$\begin{pmatrix}1&2&1\\0&-2&-1\\0&0&\frac{1}{2}\end{pmatrix}\cdot\vec{x} = \begin{pmatrix}3\\-3\\1/2\end{pmatrix}$$

$\Rightarrow$ $x_3 = 1$

$x_2 = (-3+1)/-2 = 1$

$x_1 = 0$

Suppose

eg. $A = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}l_{11} & 0\\ l_{21} & l_{22}\end{pmatrix}\cdot\begin{pmatrix}u_{11} & u_{12}\\ 0 & u_{22}\end{pmatrix}$

$$= \begin{pmatrix}0 = l_{11}\cdot u_{11} & l_{11}\cdot u_{12}\\ l_{21}\cdot u_{11} & l_{21}\cdot u_{12} + l_{22}\cdot u_{22}\end{pmatrix}$$

if $l_{11} = 0$ then $l_{11}\cdot u_{12} = 0$

similarly for u_{11}. So not possible to write

$A = L\cdot U$.

In general

$R_e \cdots | \cdots R_2 R_1 \cdot A = U$

R_i ??

$\Downarrow$

$\underbrace{R_e' \cdots R_2' R_1'}_{\text{lower triangular}} \underbrace{P}_{\text{permutation matrix / switching operation}} \cdot A = U$

- Composition of switch gives permutation of rows: if R_1 switch r_2 and r_1, R_2 switch r_2 and r_3.

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \xrightarrow{R_1} \begin{pmatrix} r_2 \\ r_1 \\ r_3 \end{pmatrix} \xrightarrow{R_2} \begin{pmatrix} r_2 \\ r_3 \\ r_1 \end{pmatrix}$$

- We can reverse the order of row operations with respect to "switching"

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \xrightarrow{R_1} \begin{pmatrix} r_1 \\ r_2+\lambda r_1 \\ r_3 \end{pmatrix} \xrightarrow{R_2} \begin{pmatrix} r_2+\lambda r_1 \\ r_1 \\ r_3 \end{pmatrix}$$

alternatively.

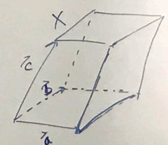
$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \xrightarrow{R_1'} \begin{pmatrix} r_2 \\ r_1 \\ r_3 \end{pmatrix} \xrightarrow{R_2'} \begin{pmatrix} r_2+\lambda r_1 \\ r_1 \\ r_3 \end{pmatrix}$$

exercise: check for general R_i

> For general $A \in M_{n\times n}$ $\exists P, L, U$ s.t.
>
> $P \cdot A = L \cdot U$
>
> "P": permutation

Determinant.

$\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$



$$Vol = (\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

determinant of $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

$$|A| := a_1 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

and $\begin{vmatrix} g & h \\ r & s \end{vmatrix} = g \cdot s - r \cdot h$.

So W
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So $Vol(X) = 0 \iff \vec{c}$ is lying in the plane of $\vec{a}$ and $\vec{b}$. (if $\vec{a}$ and $\vec{b} \neq \vec{0}$ and $\vec{a}, \vec{b}$ generate a plane)

or ($\vec{a} \cdots \vec{b}\ \vec{c}$ / $\vec{b} \cdots \vec{a}\ \vec{c}$)

$\iff \vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$.

$\iff span\langle \vec{a}, \vec{b} \rangle = span\langle \vec{a}, \vec{b}, \vec{c} \rangle$

$\iff A \cdot \vec{x} = \vec{0}$ has non-trivial solution where

$$A = \begin{pmatrix} \downarrow & \downarrow & \downarrow \\ a & b & c \end{pmatrix}$$

$\iff$ A as a linear transformation is not injective.

$\iff$ A is not surjective.

$\iff$ the echelon form of A has free variables

$\iff$ column vectors $span(\vec{a}, \vec{b}, \vec{c}) \neq \mathbb{R}^3$.

$\iff$ A is not invertible.