**Problem 1: More on derivatives and limit** Determine the derivative and second order derivatives of the following functions:

1. $f(x) = x^{100} \cdot \sin x$
2. $f(x) = \sin x \cos^2 x$
3. $f(x) = \frac{x^2-2x+1}{\sqrt{x}}$
4. $f(x) = \sin(2x)$ (Use definition of derivatives)
5. $f(x) = \sqrt{x+1}$ (Use definition of derivatives)

1. $f'(x) = x^{100} \cdot \cos x + 100 x^{99} \sin x$

$f''(x) = 100 x^{99} \cdot \cos x - x^{100} \sin x + 100 \cdot 99 \cdot x^{98} \sin x + 100 \cdot x^{99} \cdot \cos x$

$= 200 x^{99} \cdot \cos x - x^{100} \sin x + 9900 x^{98} \cdot \sin x$

2. $f'(x) = \cos^3 x - 2 \sin^2 x \cdot \cos x$

$f''(x) = -7 \sin x \cdot \cos^2 x + 2 \sin^3 x$

3. $f'(x) = \frac{3}{2} x^{\frac{1}{2}} - x^{-\frac{1}{2}} - \frac{1}{2} \cdot x^{-\frac{3}{2}}$

$f''(x) = \frac{3}{4} x^{-\frac{1}{2}} + \frac{1}{2} x^{-\frac{3}{2}} + \frac{3}{4} x^{-\frac{5}{2}}$

4. $f'(a) = \lim_{x \to a} \frac{\sin 2x - \sin 2a}{2(x-a)} \cdot 2$

$= 2 \cdot \lim_{y \to 2a} \frac{\sin y - \sin 2a}{y - 2a} = 2 \cdot \cos(2a)$

$f''(x) = -4 \sin(2x)$

5. $f'(a) = \lim_{x \to a} \frac{\sqrt{x+1} - \sqrt{a+1}}{x-a}$

$= \lim_{y \to a+1} \frac{\sqrt{y} - \sqrt{a+1}}{y-(a+1)}$

$= \frac{1}{2} \cdot (a+1)^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{4} \cdot (x+1)^{-\frac{3}{2}}$

**Problem 2: Tangent line**
Determine the tangent lines of the following functions at the given points.

1. $f(x) = \frac{x^2-1}{x}$ at $x = 2$
2. $f(x) = \frac{\sin x + \cos x}{x}$ at $x = \pi/2$
3. $f(x) = \frac{1}{x^2+1}$ at $x = 1$
4. $f(x) = \begin{cases} x^2 + 3x + 2, x < 1 \\ 5x + 1, x \geq 1 \end{cases}$ at $x = 1$

$\frac{-1}{(x^2+1)^2} \cdot 2x$

1. $y - \frac{3}{2} = \frac{5}{4} \cdot (x-2)$

2. $y - \frac{2}{\pi} = \frac{-4-2\pi}{\pi^2} \cdot (x - \frac{\pi}{2})$

3. $y - \frac{1}{2} = -\frac{1}{2}(x-1)$

4. $y - 6 = 5 \cdot (x-1)$

**Problem 3: Linear Approximation**

Use linear approximation to estimate the following function value.

1. $\sqrt{3.99999}$ $\approx 1.9999975$
2. $(1.00001)^{2022}$ $\approx 1.02022$
3. $\frac{1}{1.00001}$ $\approx 0.99999$
4. $\sin(0.2500001\pi)$ $\approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \times 0.0000001 \times \pi$
5. $\tan(0.333333\pi)$ $\approx \sqrt{3} - \frac{4}{3} \times 10^{-6} \times \pi$

**Problem 4: Application of Derivatives**

1. A bug is crawling on the $x$-axis following the function $x(t) = t^3 + 3t^2 - 9t$. Determine the velocity of the bug? When is the bug starts to turn back? (Assuming $t \geq 0$)
2. An ant is moving on the $x$-axis with $s(t) = 2t^3 - 3t^2 - 12t + 8$. Determine the time intervals when the ant is slowing down or speeding up.
3. A ball is thrown up vertically with an initial velocity $v = 100$. The height of the ball is $h(t) = -16t^2 + 100t$. Determine when the ball is highest? Determine the velocity for the ball when it hits the ground?

1. $v(t) = 3t^2 + 6t - 9$

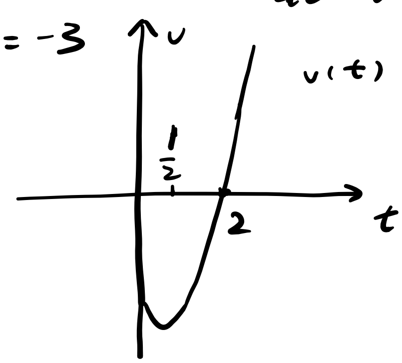
$v(t) = 0$ when $t = 1$ or $t = -3$

so it starts to turn back at $t = 1$.

2. $v(t) = 6t^2 - 6t - 12$

$[0, \frac{1}{2}] \cup [2, +\infty)$ speed up

$[\frac{1}{2}, 2]$ slow down




3. $t = \frac{25}{8}$

$h(t) = 0$ when $t = \frac{25}{4}$

$v'(t) = -32t + 100$

$v'(\frac{25}{4}) = -100$

down wards

$v = 100$