Problem 1: Chain Rule Determine the derivative and second order derivatives of the following functions:

1. $\sqrt{x^2+3x-3}$
2. $\sin(x^2)$
3. $\cos(\frac{x}{x^2+1})$
4. $\frac{1-\cos(2x)}{\sqrt{\sin x}}$
5. $(\sin 2x + \cos x^2)^{100}$
6. $\cos(\sin(\cos x))$
7. $(\frac{x}{x^2-1})^3$

Problem 2: Implicit Differentiation
Determine the derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of the functions $y = y(x)$ defined by the following equations.

1. $x^2 + y^2 = 2$ at $x = 1$, $y = 1$
2. $y^2 + xy - 2x = 0$ at $x = 1$, $y = 2$
3. $x^2 \sin(y) + y = \cos x$
4. $y^3 + x^3 - 3xy = 0$

Prob 1. 1) $f' = \frac{1}{2} \cdot (x^2+3x-3)^{-\frac{1}{2}} \cdot (2x+3)$

$$f'' = \frac{1}{2} \cdot (-\frac{1}{2}) \cdot (x^2+3x-3)^{-\frac{3}{2}} \cdot (2x+3)^2$$
$$+ (x^2+3x-3)^{-\frac{1}{2}}$$

2) $f' = \cos(x^2) \cdot 2x \qquad f'' = -\sin(x^2) \cdot 4x^2 + 2 \cdot \cos(x^2)$

3) $f' = -\sin(\frac{x}{x^2+1}) \cdot \frac{1-x^2}{(x^2+1)^2}$

$$f'' = -\cos(\frac{x}{x^2+1}) \cdot \frac{(1-x^2)^2}{(x^2+1)^4}$$
$$- \sin(\frac{x}{x^2+1}) \cdot \frac{2x \cdot (x^2-3)}{(x^2+1)^3}$$

4). $f' = \frac{\sin(2x) \cdot 2 \cdot \sqrt{\sin x} - (1-\cos 2x) \cdot \frac{1}{2} \cdot (\sin x)^{-\frac{1}{2}} \cdot \cos x}{\sin x}$

$$= 2 \cdot \sin(2x) \cdot (\sin x)^{-\frac{1}{2}} - \frac{1}{2} \cdot (1-\cos 2x) \cdot \cos x \cdot (\sin x)^{-\frac{3}{2}}$$

$$f'' = 2 \cdot \cos(2x) \cdot 2 \cdot (\sin x)^{-\frac{1}{2}} + 2\sin(2x) \cdot [-\frac{1}{2} \cdot (\sin x)^{-\frac{3}{2}}] \cdot \cos x$$
$$- \frac{1}{2} \cdot (\sin(2x) \cdot 2) \cdot \cos x \cdot (\sin x)^{-\frac{3}{2}}$$
$$- \frac{1}{2} \cdot (1-\cos 2x) \cdot [-\sin x] \cdot (\sin x)^{-\frac{3}{2}}$$
$$- \frac{1}{2} (1-\cos 2x) \cdot \cos x \cdot [-\frac{3}{2} \cdot (\sin x)^{-\frac{5}{2}} \cdot \cos x]$$

5). $f' = 100 \cdot [\sin(2x) + \cos(x^2)]^{99} \cdot [2\cos(2x) - \sin(x^2) \cdot 2x]$

$$f'' = 100 \cdot 99 \cdot [\sin(2x) + \cos(x^2)]^{98} [2\cos(2x) - 2x\sin(x^2)]^2$$
$$+ 100 \cdot [\sin(2x) + \cos(x^2)]^{99} \cdot [-4\sin(2x) - \cos(x^2) \cdot 4x^2 - 2\sin(x^2)]$$

6) $f' = \sin(\sin(\cos x)) \cdot \cos(\cos x) \cdot \sin x$

$$f'' = \cos(\sin(\cos x)) \cdot \cos^2(\cos x) \cdot \sin^2 x +$$
$$\sin(\sin(\cos x)) \cdot \sin(\cos x) \cdot \sin^2 x +$$
$$\sin(\sin(\cos x)) \cdot \cos(\cos x) \cdot \cos x$$

7) $f' = -3 \cdot \left(\frac{x}{x^2-1}\right)^2 \cdot \frac{1+x^2}{(x^2-1)^2}$

$$= -3 \cdot \frac{x^4+x^2}{(x^2-1)^4}$$

$$f'' = -3 \cdot \frac{(4x^3+2x)\cdot(x^2-1)^4 - (x^4+x^2)\cdot 4(x^2-1)^3 \cdot 2x}{(x^2-1)^8}$$

$$= -3\left[\frac{2x\cdot(2x^2+1)}{(x^2-1)^4} - \frac{8x^3\cdot(1+x^2)}{(x^2-1)^5}\right]$$

Prob 2. 1). $2x + 2y \cdot y' = 0 \Rightarrow y' = \frac{-x}{y}$

$$y'' = \frac{-1\cdot y - (-x)\cdot y'}{y^2} = -\frac{1}{y} + \frac{x}{y^2} \cdot \frac{-x}{y} = \frac{2}{-y^3}$$

$y'|_{(1,1)} = -1 \quad y''|_{(1,1)} = -2$

2) $2y \cdot y' + x \cdot y' + y - 2 = 0 \Rightarrow y' = \frac{2-y}{2y+x}$

$$y'' = \frac{-y' \cdot (2y+x) - (2-y)\cdot(2y'+1)}{(2y+x)^2}$$

$$= \frac{-x \cdot y' - 4y' - 2 + y}{(2y+x)^2} = \frac{-8}{(x+2y)^3}$$

$y'|_{(1,-2)} = -\frac{4}{3}$

$y''|_{(1,-2)} = \frac{8}{27}$

3) $2x \cdot \sin y + x^2 \cdot \cos y \cdot y' + 1 = -\sin x$

$$\Rightarrow \quad y' = \frac{-\sin x - 1 - 2x\sin y}{x^2 \cos y}$$

$$y'' = \frac{[-\cos x - 2\sin y - 2x\cos y \cdot y'] \cdot x^2\cos y - [-\sin x - 1 - 2x\sin y] \cdot [2x\cos y + x^2 \cdot (-\sin y) \cdot y']}{(x^2\cos y)^2}$$

$$= \frac{-2x^3\cos^2 y + [\sin x + 1 + 2x\sin y] \cdot x^2 \cdot (-\sin y)}{(x^2\cos y)^2} \cdot \frac{-\sin x - 1 - 2x\sin y}{x^2\cos y}$$

$$+ \frac{[-\cos x - 2\sin y] \cdot x^2\cos y + [\sin x + 1 + 2x\sin y]\, 2x\cos y}{(x^2\cos y)^2}$$

4) $3y^2 \cdot y' + 3x^2 - 3y - 3xy' = 0$

$$\Rightarrow \quad y' = \frac{y - x^2}{y^2 - x}$$

$$\Rightarrow \quad y'' = \frac{[y' - 2x] \cdot [y^2 - x] - [y - x^2] \cdot (2yy' - 1)}{(y^2 - x)^2}$$

$$= \frac{(y^2 - x) - 2y \cdot (y - x^2)}{(y^2 - x)^2} \cdot \frac{y - x^2}{y^2 - x}$$

$$+ \frac{-2x \cdot (y^2 - x) + (y - x^2)}{(y^2 - x)^2}$$