Problem 1 : Know the Curves
Describe the Shape of the Following Curve $\vec{x}(t)$:

1. $\begin{pmatrix} 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ line

2. $\begin{pmatrix} t \\ t^2 \end{pmatrix}$ parabola

3. $\begin{pmatrix} 3t \\ 4 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} t+1 \\ t-1 \\ 2t \end{pmatrix}$ line

4. $\begin{pmatrix} 3\cos \pi t \\ 3 \sin \pi t \end{pmatrix}$ circle

5. $\begin{pmatrix} \cos \pi t \\ \sin \pi t \\ t \end{pmatrix}$ helix

6. $\begin{pmatrix} 1 + 3\cos \pi t \\ -2 + 3 \sin \pi t \end{pmatrix}$ circle

Problem 2: Tangent Vector and Tangent Line
Compute the tangent vector of following curves and write the parametrization of the tangent line at the given point:

1. $\begin{pmatrix} t^2 \\ t^3 \\ t^4 \end{pmatrix}$, $t = 1$

2. $\begin{pmatrix} 2\theta - 2\sin\theta \\ 2 - 2\cos\theta \end{pmatrix}$, $\theta = \pi$

1. $\vec{x}'(t) = \begin{pmatrix} 2t \\ 3t^2 \\ 4t^3 \end{pmatrix}$

2. $\vec{x}'(\theta) = \begin{pmatrix} 2 - 2\cos\theta \\ 2\sin\theta \end{pmatrix}$

$\vec{l}(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$\vec{l}(t) = \begin{pmatrix} 2\pi \\ 4 \end{pmatrix} + t \cdot \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

since $\vec{x}(1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\vec{x}'(1) = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

Problem 3: Computing Arc Length
Compute the arc length between given points.

1. $\begin{pmatrix} \cos \pi t \\ \sin \pi t \\ t \end{pmatrix}$ from $t = 0$ to $t = \pi$

2. $\begin{pmatrix} e^t \cos t \\ e^t \sin t \end{pmatrix}$ from $t = 0$ to $t = \pi$

3. $\begin{pmatrix} t - \sin t \\ 1 - \cos t \end{pmatrix}$ from $t = 0$ to $t = \pi$

1. $\vec{x}'(t) = \begin{pmatrix} -\pi \sin \pi t \\ \pi \cos \pi t \\ 1 \end{pmatrix}$ $\quad \|\vec{x}'(t)\| = \sqrt{\pi^2+1}$ $\quad \int_0^{\pi} \sqrt{\pi^2+1}\, dt = \pi \cdot \sqrt{\pi^2+1}$

2. $\vec{x}'(t) = \begin{pmatrix} e^t(\cos t - \sin t) \\ e^t \cdot (\sin t + \cos t) \end{pmatrix}$ $\quad \|\vec{x}'(t)\| = \sqrt{2} \cdot e^t$ $\quad \int_0^{\pi} \sqrt{2}\, e^t\, dt = \sqrt{2} \cdot (e^{\pi} - 1)$

3. $\vec{x}'(t) = \begin{pmatrix} 1 - \cos t \\ \sin t \end{pmatrix}$ $\quad \|\vec{x}'(t)\| = \sqrt{2 - 2\cos\theta}$ $= 2 \sin \frac{\theta}{2}$ $\quad \int_0^{\pi} 2 \sin \frac{\theta}{2}\, d\theta = 4$

Problem 4: Curvature
Compute the curvature vector for the following curves:

1. $\begin{pmatrix} \cos \pi t \\ \sin \pi t \\ t \end{pmatrix}$

2. $\begin{pmatrix} e^t \cos t \\ e^t \sin t \end{pmatrix}$

1. $\vec{T}(t) = \dfrac{\vec{x}'(t)}{\|\vec{x}'(t)\|} = \dfrac{1}{\sqrt{\pi^2+1}} \cdot \begin{pmatrix} -\pi \sin \pi t \\ \pi \cos \pi t \\ 1 \end{pmatrix}$

$\vec{k}(t) = \dfrac{\vec{T}'(t)}{\|\vec{x}'(t)\|} = \dfrac{1}{\sqrt{\pi^2+1}} \cdot \dfrac{1}{\sqrt{\pi^2+1}} \cdot \begin{pmatrix} -\pi^2 \cos \pi t \\ -\pi^2 \sin \pi t \\ 0 \end{pmatrix}$

$= \dfrac{-\pi^2}{\pi^2+1} \begin{pmatrix} \cos \pi t \\ \sin \pi t \\ 0 \end{pmatrix}$

2. $\vec{T}(t) = \dfrac{\vec{x}'(t)}{\|\vec{x}'(t)\|} = \dfrac{1}{\sqrt{2}} \cdot \begin{pmatrix} \cos t - \sin t \\ \sin t + \cos t \end{pmatrix}$

$\vec{k}(t) = \dfrac{\vec{T}'(t)}{\|\vec{x}'(t)\|} = \dfrac{1}{\sqrt{2}\, e^t} \cdot \dfrac{1}{\sqrt{2}} \cdot \begin{pmatrix} -\sin t - \cos t \\ \cos t - \sin t \end{pmatrix}$

$= \dfrac{1}{2e^t} \begin{pmatrix} -\sin t - \cos t \\ \cos t - \sin t \end{pmatrix}$