MATH 234 WORKSHEET 4 FEB 13 2016, TA: JIUYA WANG

Problem 1: Limit for Two Variable Function

Determine if the following functions have a limit as (x, y) approaches $(0, 0)$:

1. $f(x,y) = \frac{xy}{x^2+y^2}$ No
2. $f(x,y) = \frac{x^2}{x^2+y^2}$ No
3. $f(x,y) = \frac{x^2}{\sqrt{x^2+y^2}}$ Yes

1. Restrict the function to $y=kx$

$f(x,y) = \frac{x \cdot kx}{x^2+k^2x^2} = \frac{k}{1+k^2}$

so $\lim_{x\to 0} f(x,kx) = \frac{k}{1+k^2}$

Choosing different k gives different limit

So there is no limit for $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

2. Restrict the f to $y=kx$

so $f(x,y) = \frac{1}{1+k^2}$

Choosing different k gives different limit. So no limit.

3. $f(x,y) = \frac{r^2\cos^2\theta}{r} = r\cos^2\theta \le r$

so $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} r\cos^2\theta = 0$

Problem 2 : Tangent Plane/Linear Approximation

For following surfaces:

1) compute the partial derivatives;
2) write up the differential df;
3) find out the tangent plane at the given point;
4) what is the linear approximation at the given point;
4) what is the normal vector of the plane you find out.

1. $z = xy^2$; $x = 2, y = 1$
2. $z = \frac{xy}{x+y}$; $x = 3, y = 1$
3. $x^2 + y^2 + z^2 = 3$; $x = 1, y = 1, z = 1$; $x = 1, y = 1, z = -1$

1. 1) $\frac{\partial z}{\partial x} = y^2 \quad \frac{\partial z}{\partial y} = 2xy$

2) $df = y^2dx + 2xy\,dy$

3) $df = dx + 4dy$ @ $(2,1)$

$z-2 = (x-2) + 4\cdot(y-1)$

4) $z \approx (x-2) + 4\cdot(y-1) + 2$

5) $\vec{n} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$

2. 1) $\frac{\partial z}{\partial x} = \frac{y^2}{(x+y)^2} \quad \frac{\partial z}{\partial y} = \frac{x^2}{(x+y)^2}$

2) $df = \frac{y^2}{(x+y)^2}dx + \frac{x^2}{(x+y)^2}dy$

3) $df = \frac{1}{16}dx + \frac{9}{16}dy$

$z - \frac{3}{4} = \frac{1}{16}(x-3) + \frac{9}{16}(y-1)$

4) $z \approx \frac{3}{4} + \frac{1}{16}(x-3) + \frac{9}{16}(y-1)$

5) $\vec{n} = \begin{pmatrix} \frac{1}{16} \\ \frac{9}{16} \\ -1 \end{pmatrix}$

3. 1) $2x\,dx + 2y\,dy + 2z\,dz = 0$

$dz = -\frac{x}{z}dx - \frac{y}{z}dy \Rightarrow \frac{\partial z}{\partial x} = -\frac{x}{z} \quad \frac{\partial z}{\partial y} = -\frac{y}{z}$

[Here if you feel not comfortable, just compute $z = \sqrt{3-x^2-y^2}$ $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ and $z = -\sqrt{3-x^2-y^2}$]

3) $dz = dx + dy$ at $(1,1,-1)$

$(z+1) = (x-1) + (y-1)$

$x + y - z = 3$

4) $z \approx x + y - 3$

5) $\vec{n} = (1, 1, -1)$

For $(1,1,1)$

$dz = -dx - dy$

$z - 1 = -(x-1) - (y-1)$

$x + y + z = 1$

$z \approx 1 - x - y$

$\vec{n} = (1, 1, 1)$

Problem 3: Chain Rule

Firstly use chain rule to compute $\frac{df}{dt}$, and then evaluate $\frac{df}{dt}$ at given point:

1. $f(x,y) = x^2y^3 + x^3y^2$; $x(t) = t^2 + t$, $y(t) = e^t$; $t = 0$;
2. $f(x,y) = x^2 + y^2$; $x(t) = \cos t$, $y(t) = 2\sin t$; $t = \frac{\pi}{2}$;
3. $f(x,y,z) = xyz$; $x(t) = \ln t$, $y(t) = e^t$, $z(t) = \frac{1}{t}$; $t = 2$;

1. $\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$

$= (2xy^3 + 3x^2y^2) \cdot (2t+1) + (3y^2x^2 + 2yx^3) \cdot e^t$

$\left.\frac{dt}{dt}\right|_{t=0} = 0$

2. $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = 2x \cdot (-\sin t) + 2y \cdot (2\cos t)$

$\left.\frac{df}{dt}\right|_{t=\frac{\pi}{2}} = 2 \cdot 0 \cdot -1 + 2 \cdot 2 \cdot 0 = 0$

(because $x(\frac{\pi}{2}) = 0$ $y(\frac{\pi}{2}) = 2$)

3. $\frac{dt}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} = yz \cdot \frac{1}{t} + xz \cdot e^t + xy \cdot \frac{-1}{t^2}$

$\left.\frac{df}{dt}\right|_{t=2} = e^2 \cdot \frac{1}{2} \cdot \frac{1}{2} + \ln 2 \cdot \frac{1}{2} \cdot e^2 + \ln 2 \cdot e^2 \cdot \frac{-1}{4} = e^2 \cdot \frac{1}{2} \cdot \ln 2$