Problem 1 : Implicit Function Derivative
For the following equations:
1. view $z = z(x, y)$ from the equations, then use implicit function method to compute the partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$;
2. view $y = y(x, z)$ from the equations, then use implicit function method to compute the partial derivatives $\frac{\partial y}{\partial x}, \frac{\partial y}{\partial z}$;
2. at which points the partial derivatives $\frac{\partial z}{\partial x}$ are not defined? at which points the partial derivatives $\frac{\partial y}{\partial x}$ are not defined?

1. $x^2 + y^2 + z^2 = 1$
2. $x^2y + y^2z + z^2x = 0$
3. $xy + yz + xz = -1$

1. $F(x, y, z) = x^2 + y^2 + z^2 - 1$

$F_x = 2x \quad F_y = 2y \quad F_z = 2z$

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{x}{z}$

$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{y}{z}$

Similarly,

$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{x}{y}$

$\frac{\partial y}{\partial z} = -\frac{F_z}{F_y} = -\frac{z}{y}$

$\frac{\partial z}{\partial x}$ is not defined at $F_z = 2z = 0$

$\frac{\partial y}{\partial x}$ is not defined at $F_y = 2y = 0$

2. $F(x, y, z) = x^2y + y^2z + z^2x$

$F_x = 2xy + z^2$ $\quad \frac{\partial z}{\partial x} = -\frac{z^2 + 2xy}{y^2 + 2xz}$

$F_y = x^2 + 2yz$ $\quad \frac{\partial z}{\partial y} = -\frac{x^2 + 2yz}{y^2 + 2xz}$

$F_z = y^2 + 2xz$

$\frac{\partial y}{\partial x} = -\frac{2xy + z^2}{x^2 + 2yz}$ $\quad \frac{\partial y}{\partial z} = -\frac{y^2 + 2xz}{x^2 + 2yz}$

$\frac{\partial z}{\partial x}$ is not defined at $y^2 + 2xz = 0$

$\frac{\partial y}{\partial x}$ is not defined at $x^2 + 2yz = 0$

Problem 2: Chain Rule Use chain rule to compute $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ (and $\frac{\partial f}{\partial w}$):

1. $f(x, y) = \sin(x^2 + y^2)$; $x(u, v) = u^2 - v^2$; $y(u, v) = 2uv$;
2. $f(x, y) = x^2y$; $x(u, v) = \sin uv$, $y(u, v) = e^{uv}$;
3. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$; $x(u, v) = u \sin v \cos w$, $y(u, v) = u \sin v \sin w$, $z(u, v) = u \cos v$;

3. $F(x, y, z) = xy + yz + xz + 1$

$F_x = y + z$ $\quad \frac{\partial z}{\partial x} = -\frac{y + z}{x + y}$ $\quad \frac{\partial z}{\partial y} = -\frac{x + z}{x + y}$ $\quad \frac{\partial z}{\partial x}$ is not defined at $x + y = 0$

$F_y = x + z$

$F_z = x + y$ $\quad \frac{\partial y}{\partial x} = -\frac{y + z}{x + z}$ $\quad \frac{\partial y}{\partial z} = -\frac{x + y}{x + z}$ $\quad \frac{\partial y}{\partial z}$ is not defined at $x + z = 0$

1. $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} = 2x \cdot \cos(x^2+y^2) \cdot 2u + 2y \cdot \cos(x^2+y^2) \cdot (2v)$

$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} = 2x \cdot \cos(x^2+y^2) \cdot (-2v) + 2y \cdot \cos(x^2+y^2) \cdot (2u)$

2. $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} = 2xy \cdot \cos(uv) \cdot v + x^2 \cdot e^{uv} \cdot v$

$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} = 2xy \cdot \cos(uv) \cdot u + x^2 \cdot e^{uv} \cdot u$

3. $\frac{\partial f}{\partial u} = f_x \cdot X_u + f_y \cdot Y_u + f_z \cdot Z_u = \frac{x}{\sqrt{x^2+y^2+z^2}} \cdot \sin v \cos w + \frac{y}{\sqrt{x^2+y^2+z^2}} \cdot \sin v \sin w + \frac{z}{\sqrt{x^2+y^2+z^2}} \cdot \cos v$

$\frac{\partial f}{\partial v} = \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot [x \cdot u \cos w \cdot \cos v + y \cdot u \cdot \sin w \cdot \cos v + z \cdot u \cdot (-\sin v)]$

$\frac{\partial f}{\partial w} = \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot [x \cdot u \sin v \cdot (-\sin w) + y \cdot u \sin v \cdot \cos w]$