Problem 1 : Local Max/Min/Saddle with second order derivative test
For following functions:
1. Compute first order and second order derivatives;
2. Compute the critical points;
3. Determine the local behaviors at critical points using second order derivative test.

1. $f(x,y) = x^2 + 4y^2 - 2x + 8y - 1$;
2. $f(x,y) = (x-y)(xy-4)$;
3. $f(x,y) = y^2 - 18x^2 + x^4$;

1. $f_x = 2x - 2$ $f_{xx} = 2$ $f_{xy} = 0$

$f_y = 8y + 8$ $f_{yy} = 8$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{matrix} x = 1 \\ y = -1 \end{matrix}$ c.p. : $(1,-1)$

$\begin{cases} \Delta = f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot 8 - 0 = 16 > 0 \\ f_{xx} = 2 > 0 \end{cases}$

$\Rightarrow$ local min

2. $f_x = 2xy - 4 - y^2$ $f_{xx} = 2y$ $f_{xy} = 2x - 2y$

$f_y = x^2 - 2xy + 4$ $f_{yy} = -2x$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 2xy - 4 - y^2 = 0 \\ 2xy - 4 - x^2 = 0 \end{cases} \Rightarrow x^2 = y^2$

if $x = y$, $\Rightarrow x = y = 2$ or -2

if $x = -y$ $\Rightarrow$ no solution

so c.p. : $(2,2)$ or $(-2,-2)$

at $(2,2)$ $f_{xx} = 4$ $f_{yy} = -4$ $f_{xy} = 0$

$\Delta = 4 \cdot (-4) - 0 < 0 \Rightarrow$ saddle pt

at $(-2,-2)$ $f_{xx} = -4$ $f_{yy} = 4$ $f_{xy} = 0$

$\Delta = (-4) \cdot 4 - 0 < 0 \Rightarrow$ saddle pt.

3. $f_x = -36x + 4x^3$ $f_{xx} = -36 + 12x^2$, $f_{xy} = 0$

$f_y = 2y$ $f_{yy} = 2$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \text{ or } 3 \text{ or } -3 \\ y = 0 \end{cases}$

c.p : $(0,0)$ $(3,0)$ $(-3,0)$

at $(0,0)$: $f_{xx} = -36$ $f_{xy} = 0$ $f_{yy} = 2$

$\Delta = -36 \cdot 2 - 0 < 0$ saddle pt

at $(3,0)$: $f_{xx} = 72$ $f_{xy} = 0$ $f_{yy} = 2$

$\begin{cases} \Delta = 72 \cdot 2 - 0 > 0 \\ f_{xx} > 0 \end{cases} \Rightarrow$ local min

at $(-3,0)$: $f_{xx} = 72$ $f_{xy} = 0$ $f_{yy} = 2$

$\begin{cases} \Delta = 72 \cdot 2 - 0 > 0 \\ f_{xx} > 0 \end{cases} \Rightarrow$ local min

Problem 2 : Other Local Max/Min/Saddle
For following functions:
1. Compute the critical points;
2. Determine the local behaviors at critical points.

1. $f(x,y) = y^2 + x^4$;
2. $f(x,y) = y^2 - x^4$;
3. $f(x,y) = x^3 - xy^2$;

1. $f_x = 4x^3$ $f_{xx} = 12x^2$ $f_{xy} = 0$

$f_y = 2y$ $f_{yy} = 2$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$ c.p. : $(0,0)$

$\Delta = 0$ at $(0,0)$ inconclusive.

$f(x,y) = y^2 + x^4 > f(0,0)$ when $(x,y) \neq (0,0)$

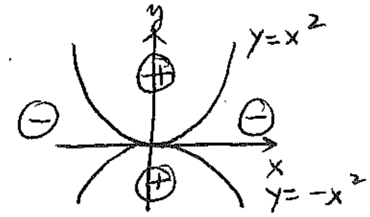
so $(0,0)$ local min

2. $f_x = -4x^3$ $f_{xx} = -12x^2$ $f_{xy} = 0$

$f_y = 2y$ $f_{yy} = 2$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$ c.p. : $(0,0)$ $\Delta = 0$ inconclusive.

$(y^2 - x^4) = (y + x^2) \cdot (y - x^2)$



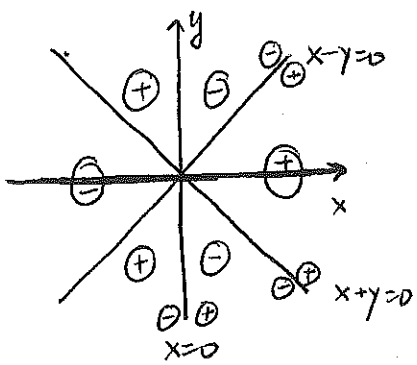
So neither max nor min!

3. $f_x = 3x^2 - y^2$ $\quad f_{xx} = 6x$ $\quad f_{xy} = -2y$

$f_y = -2xy$ $\quad f_{yy} = -2x$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$ $\quad$ c.p. : $(0, 0)$ $\quad \Delta = 0$ $\quad$ inconclusive

$f(x,y) = x \cdot (x+y) \cdot (x-y)$



Six regions, alternating sign.

so neither max nor min.