Problem 1 : Lagrange Multiplier
Find all the optimal points for the following questions by Lagrange Multiplier.

1. Page 104 Problem 1
2. Page 104 Problem 5 a
3. Page 104 Problem 8

Problem 2 : Iterated Integral
Compute the following iterated integral:

1. $\int_0^1 \int_0^4 x dy dx$; $\int_0^1 \int_0^4 x dx dy$;
2. $\int_{-1}^1 \int_0^{x^2} (x^2 + y^2) dy dx$;
3. $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x dy dx$;

Problem 3 : Domain is important!
For all the following iterated integral:
1. Draw the domain D for the corresponding double integral;
2. Rewrite your double integral back to iterated integral with the other order.

1. $\int_0^1 \int_0^x f(x,y) dy dx$;
2. $\int_0^1 \int_0^{x^2} f(x,y) dy dx$;
3. $\int_0^1 \int_{x^2}^x f(x,y) dy dx$;
4. $\int_0^1 \int_{x^2}^{\sqrt{x}} f(x,y) dy dx$;

Problem 4 : Double Integral

For the following double integral:
1. Draw the domain and pick one order to write the double integral to iterated integral;
2. Compute the iterated integral;

1. $\iint_D (1+x)dA$; $D = \{(x,y) : 0 \le x \le 2, -x \le y \le x\}$
2. $\iint_D dA$; $D = \{(x,y) : 0 \le x \le 1, 0 \le y \le x\}$
   (Q: what are you computing when the integrand is 1 ?)
3. $\iint_D y\sin(x^2)dA$; $D = \{(x,y) : 0 \le y \le 1, y^2 \le x \le 1\}$
   Hint: can you integrate? How about changing the order as you did in Problem 2?
4. $\iint_D \frac{2}{1-x^2}dA$; D is the triangle bounded by the y axis, $y = 1$, and $y = x$

Problem 5 : Polar Coordinates

When the domain is circle (or something similar), using polar coordinates is convenient to compute integral.

1. $\iint_D dA$; D is unit circle;
2. $\iint_D (x^2+y^2)dA$; D is upper half circle;
3. Compute the volume of the cone: the region bounded by $z = \sqrt{x^2+y^2}$ and $z = 1$

Lagrange Multiplier:

1. $\begin{cases} f(x,y) = xy \\ g(x,y) = x^2 + \frac{1}{4}y^2 = 1 \end{cases}$ for c.p: $\begin{cases} \nabla f = \lambda \cdot \nabla g \\ x^2 + \frac{1}{4}y^2 = 1 \end{cases} \Leftrightarrow \begin{cases} y = \lambda \cdot 2x & ① \\ x = \lambda \cdot \frac{1}{2} y & ② \\ x^2 + \frac{1}{4}y^2 = 1 & ③ \end{cases}$

$\nabla g = 0 \Rightarrow x = y = 0$ but $g(0,0) \neq 1$

① → ② $x = \lambda \cdot \frac{1}{2} \cdot \lambda \cdot 2x = \lambda^2 x \Rightarrow (\lambda^2 - 1) \cdot x = 0 \Rightarrow x = 0$ or $\lambda = \pm 1$

if $x = 0$ then from ① $y = 0$ then ③ does not hold.

if $\lambda = 1$. $y = 2x$ by ①. plug in ③ $x^2 + \frac{1}{4} \cdot 4x^2 = 2x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$

$\left(\frac{1}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$ $\left(\frac{-1}{\sqrt{2}}, \frac{-2}{\sqrt{2}}\right)$

if $\lambda = -1$ $y = -2x$ by ① plug in ③ $x = \pm \frac{1}{\sqrt{2}}$ $\left(\frac{1}{\sqrt{2}}, -\frac{2}{\sqrt{2}}\right)$ $\left(\frac{-1}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$

so there're 4 c.p. The minimal value for $f(x,y) = xy$ is -1 at. $\left(\frac{\pm 1}{\sqrt{2}}, \frac{\mp 2}{\sqrt{2}}\right)$

2. distance^2: $f(x,y) = (x-2)^2 + (y-1)^2 + (z-4)^2$ $\nabla g = 0 \Rightarrow$ no such x, y, z

$g(x,y) = 2x - y + 3z = 1$

for c.p. $\begin{cases} \nabla f = \lambda \cdot \nabla g \\ 2x - y + 3z = 1 \end{cases} \Leftrightarrow \begin{cases} 2(x-2) = \lambda \cdot 2 & ① \\ 2(y-1) = -\lambda & ② \\ 2(z-4) = 3\lambda & ③ \\ 2x - y + 3z = 1 & ④ \end{cases}$ by ① $x = \lambda + 2$ ② $y = -\frac{\lambda}{2} + 1$ ③ $z = \frac{3}{2}\lambda + 4$ plug all in ④ $\Rightarrow \lambda = -2$

c.p: $x = 0$ $y = 2$ $z = 1$. The minimal distance is $\sqrt{(0-2)^2 + (2-1)^2 + (1-4)^2} = \sqrt{14}$.

3. $\frac{1}{2}$·Surface Area: $f(x,y,z) = xy + yz + xz$

Volume: $g(x,y,z) = xyz = \frac{1}{2}$ $\nabla g = 0 \Rightarrow g = 0 \neq \frac{1}{2}$

for c.p: $\begin{cases} \nabla f = \lambda \nabla g \\ xyz = \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} y + z = \lambda \cdot yz & ① \\ x + z = \lambda \cdot xz & ② \\ x + y = \lambda \cdot xy & ③ \\ xyz = \frac{1}{2} & ④ \end{cases}$ ①×x $\Rightarrow xy + yz = \lambda xyz$ compare with ②×y ③×z we have $xy + yz = xy + yz = xz + yz = \lambda xyz$

$\Rightarrow xy = yz = xz = \frac{1}{2}\lambda xyz = \frac{1}{2}\lambda \cdot \frac{1}{2} = \frac{\lambda}{4}$ plug in ① ② ③ $y + z = x + z = x + y = \frac{\lambda^2}{4}$

$\Rightarrow x = y = z = \frac{\lambda^2}{8}$ plug in ④. $\frac{\lambda^6}{8^3} = \frac{1}{2} \Rightarrow \lambda = 2^{\frac{4}{3}}$ $x = y = z = 2^{-\frac{1}{3}}$

The shape of the box should be. Length × Wide × Height $= 2^{-\frac{1}{3}} \times 2^{-\frac{1}{3}} \times 2^{-\frac{1}{3}}$

Worksheet 9.

P1. 1. $\int_0^1 \int_0^4 x\,dy\,dx = \int_0^1 \left(\int_0^4 x\,dy \right) dx = \int_0^1 (x \cdot y)\Big|_{y=0}^{y=4} dx$

$= \int_0^1 4x\,dx = 2x^2\Big|_0^1 = 2$

2. $\int_0^1 \int_0^4 x \cdot dx\,dy = \int_0^1 \left(\int_0^4 x \cdot dx \right) dy = \int_0^1 \left(\frac{x^2}{2}\Big|_{x=0}^{x=4} \right) dy$

$= \int_0^1 8\,dy = 8y\Big|_0^1 = 8$

3. $\int_{-1}^1 \int_0^{x^2} (x^2 + y^2)\,dy\,dx = \int_{-1}^1 \left(x^2 \cdot y + \frac{y^3}{3}\right)\Big|_{y=0}^{y=x^2} dx$

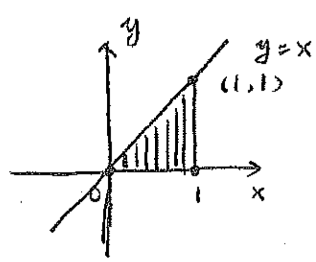
$= \int_{-1}^1 \left(x^4 + \frac{x^6}{3}\right) dx = \left(\frac{x^5}{5} + \frac{x^7}{21}\right)\Big|_{-1}^1 = \frac{52}{105}$

4. $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\,dy\,dx = \int_0^1 (x \cdot y)\Big|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dx$

$= \int_0^1 2x \cdot \sqrt{1-x^2}\,dx \overset{u=1-x^2}{=\!=} -\int_1^0 \sqrt{u}\,du = \frac{2}{3} \cdot u^{\frac{3}{2}}\Big|_0^1$

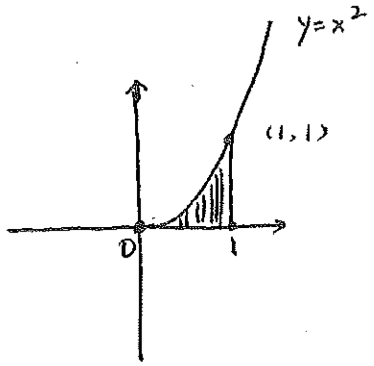
$= \frac{2}{3}$

P2. 1.



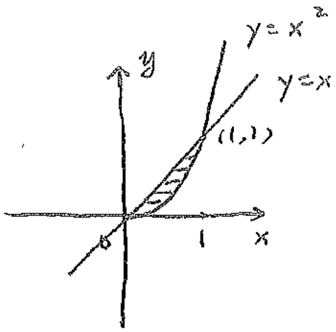
$\int_0^1 \int_y^1 f(x,y)\,dx\,dy$

2.



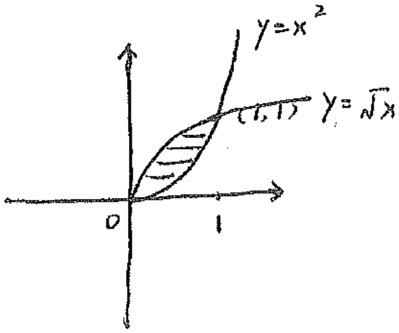
$\int_0^1 \int_{\sqrt{y}}^1 f(x,y)\,dx\,dy$

3.



$$\int_0^1 \int_y^{\sqrt{y}} f(x,y)\,dx\,dy$$

4.

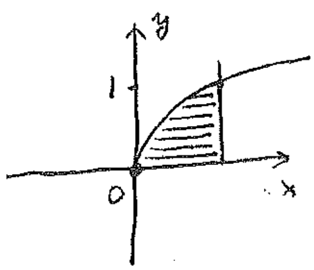


$$\int_0^1 \int_{y^2}^{\sqrt{y}} f(x,y)\,dx\,dy$$

P3. 1. $$\int_0^2 \int_{-x}^{x} (1+x)\,dy\,dx = \int_0^2 (y+xy)\Big|_{y=-x}^{y=x} dx = \int_0^2 (2x+2x^2)\,dx = \frac{28}{3}$$

2. $$\int_0^1 \int_0^x 1 \cdot dy\,dx = \int_0^1 y\Big|_{y=0}^{y=x} dx = \int_0^1 x\,dx = \frac{1}{2}$$

3.

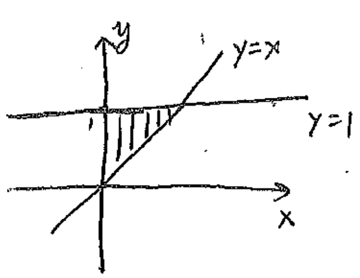


$$\int_0^1 \int_0^{\sqrt{x}} y \sin x^2\,dy\,dx$$

$$= \int_0^1 \sin x^2 \cdot \frac{y^2}{2}\Big|_0^{y=\sqrt{x}} dx = \int_0^1 \sin x^2 \cdot \frac{x}{2}\,dx$$

$$= \frac{1}{4}\cdot(-\cos x^2)\Big|_0^1 = \frac{1}{4}\cdot(1-\cos 1)$$

4.



$$\int_0^1 \int_x^1 \frac{2}{1-x^2}\,dy\,dx$$

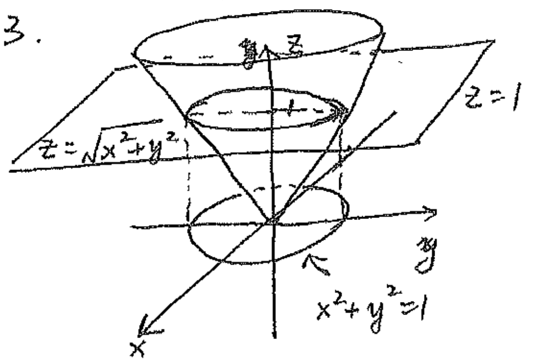
$$= \int_0^1 \frac{2}{1-x^2}\cdot(1-x)\cdot dx = \int_0^1 \frac{2}{x+1}\,dx$$

$$= 2\ln(x+1)\Big|_0^1 = 2\ln 2$$

P4. 1. $$\int_0^1\int_0^{2\pi} r\cdot d\theta\, dr = \int_0^1 2\pi r\cdot dr = \pi r^2\Big|_0^1 = \pi$$

2. $$\int_0^{\pi}\int_0^1 r^2\cdot r\,dr\,d\theta = \int_0^{\pi} \frac{r^4}{4}\Big|_0^1 d\theta = \int_0^{\pi} \frac{1}{4}\, d\theta = \frac{\pi}{4}$$

3.



The volume is computed by taking the unit circle D as the basis. for each point in D. the corresponding height is $1-z=1-r$

So $$V=\int_0^{2\pi}\int_0^1 (1-r)\cdot r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{6}\cdot d\theta = \frac{\pi}{3}$$